Loop Anomalies in the Causal Approach

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Abstract

We consider gauge models in the causal approach and prove that the loop contributions to the chronological products do not produce anomalies, at least in the second and third order of the perturbation theory. The axial anomaly is null if we impose a stronger condition, namely that the chronological products are coboundaries, up to super-renormalizable terms.

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1 Introduction

The general framework of perturbation theory consists in the construction of the chronological products such that Bogoliubov axioms are verified [1], [5], [3]; for every set of Wick monomials $W_1(x_1), \dots, W_n(x_n)$ acting in some Fock space $\mathcal{H}$ one associates the operator-valued distributions $T^{W_1,\dots,W_n}(x_1, \dots, x_n)$ called chronological products; it will be convenient to use another notation: $T(W_1(x_1), \dots, W_n(x_n))$. The construction of the chronological products can be done recursively according to Epstein-Glaser prescription [5], [6] (which reduces the induction procedure to a distribution splitting of some distributions with causal support) or according to Stora prescription [9] (which reduces the renormalization procedure to the process of extension of distributions). These products are not uniquely defined but there are some natural limitation on the arbitrariness. If the arbitrariness does not grow with n we have a renormalizable theory. An equivalent point of view uses retarded products [12].

Gauge theories describe particles of higher spin. Usually such theories are not renormalizable. However, one can save renormalizablility using ghost fields. Such theories are defined in a Fock space $\mathcal{H}$ with indefinite metric, generated by physical and un-physical fields (called *ghost fields*). One selects the physical states assuming the existence of an operator Q called *gauge charge* which verifies $Q^2 = 0$ and such that the *physical Hilbert space* is by definition $\mathcal{H}_{\rm phys} \equiv Ker(Q)/Im(Q)$. The space $\mathcal{H}$ is endowed with a grading (usually called *ghost number*) and by construction the gauge charge is raising the ghost number of a state. Moreover, the space of Wick monomials in $\mathcal{H}$ is also endowed with a grading which follows by assigning a ghost number to every one of the free fields generating $\mathcal{H}$. The graded commutator d_Q of the gauge charge with any operator A of fixed ghost number

$$d_Q A = [Q, A] \tag{1.1}$$

is raising the ghost number by a unit. It means that d_Q is a co-chain operator in the space of Wick polynomials. From now on $[\cdot, \cdot]$ denotes the graded commutator.

A gauge theory assumes also that there exists a Wick polynomial of null ghost number $T(x)$ called *the interaction Lagrangian* such that

$$[Q, T] = i\partial_\mu T^\mu \tag{1.2}$$

for some other Wick polynomials T^μ. This relation means that the expression T leaves invariant the physical states, at least in the adiabatic limit. Indeed, if this is true we have:

$$T(f)\ \mathcal{H}_{\rm phys}\ \subset\ \ \mathcal{H}_{\rm phys} \tag{1.3}$$

up to terms which can be made as small as desired (making the test function f flatter and flatter). In all known models one finds out that there exists a chain of Wick polynomials $T^\mu,\ T^{\mu\nu},\ T^{\mu\nu\rho}, \dots$ such that:

$$[Q, T] = i\partial_\mu T^\mu, \quad [Q, T^\mu] = i\partial_\nu T^{\mu\nu}, \quad [Q, T^{\mu\nu}] = i\partial_\rho T^{\mu\nu\rho}, \dots \tag{1.4}$$

It so happens that for all these models the expressions $T^{\mu\nu},\ T^{\mu\nu\rho}, \dots$ are completely antisymmetric in all indexes; it follows that the chain of relation stops at the step 4 (if we work in

four dimensions). We can also use a compact notation T^I where I is a collection of induces $I = [\nu_1, \ldots, \nu_p]$ $(p = 0, 1, \ldots,)$ and the brackets emphasize the complete antisymmetry in these induces. All these polynomials have the same canonical dimension

$$\omega(T^I) = \omega_0, \ \forall I \tag{1.5}$$

and because the ghost number of $T \equiv T^{\emptyset}$ is supposed null, then we also have:

$$gh(T^I) = |I|. \tag{1.6}$$

One can write compactly the relations (1.4) as follows:

$$d_Q T^I = i \ \partial_\mu T^{I\mu}. \tag{1.7}$$

For concrete models the equations (1.4) can stop earlier: for instance in the Yang-Mills case we have $T^{\mu\nu\rho} = 0$ and in the case of gravity $T^{\mu\nu\rho\sigma} = 0$.

Now we can construct the chronological products

$$T^{I_1,\ldots,I_n}(x_1, \ldots, x_n) \equiv T(T^{I_1}(x_1), \ldots, T^{I_n}(x_n)) \tag{1.8}$$

according to the recursive procedure. We say that the theory is gauge invariant in all orders of the perturbation theory if the following set of identities generalizing (1.7):

$$d_Q T^{I_1,\ldots,I_n} = i \sum_{l=1}^{n} (-1)^{s_l} \frac{\partial}{\partial x_l^\mu} T^{I_1,\ldots,I_l\mu,\ldots,I_n} \tag{1.9}$$

are true for all $n \in \mathbb{N}$ and all $I_1, \ldots, I_n$. Here we have defined

$$s_l \equiv \sum_{j=1}^{l-1} |I|_j. \tag{1.10}$$

In particular, the case $I_1 = \ldots = I_n = \emptyset$ it is sufficient for the gauge invariance of the scattering matrix, at least in the adiabatic limit: we have the same argument as for relation (1.3).

Such identities can be usually broken by *anomalies* i.e. expressions of the type $A^{I_1,\ldots,I_n}$ which are quasi-local and might appear in the right-hand side of the relation (1.9). In a previous paper we have emphasized the cohomological structure of this problem [8]. We consider a *cochain* to be an ensemble of distribution-valued operators of the form $C^{I_1,\ldots,I_n}(x_1, \ldots, x_n), \ n = 1, 2, \cdots$ (usually we impose some supplementary symmetry properties) and define the derivative operator δ according to

$$(\delta C)^{I_1,\ldots,I_n} = \sum_{l=1}^{n} (-1)^{s_l} \frac{\partial}{\partial x_l^\mu} C^{I_1,\ldots,I_l\mu,\ldots,I_n}. \tag{1.11}$$

We can prove that

$$\delta^2 = 0. \tag{1.12}$$

Next we define

$$s = d_Q - i\delta, \qquad \bar{s} = d_Q + i\delta \tag{1.13}$$

and note that

$$s\bar{s} = \bar{s}s = 0. \tag{1.14}$$

We call *relative cocycles* the expressions C verifying

$$sC = 0 \tag{1.15}$$

and a *relative coboundary* an expression C of the form

$$C = \bar{s}B. \tag{1.16}$$

The relation (1.9) is simply the cocycle condition

$$sT = 0. \tag{1.17}$$

Gauge theories have been intensively studied in another formalism based on functional integrations and Green functions. There is no proof of the equivalence between the functional formalism and the causal formalism which we use here. A supplementary problem in the functional formalism is that the Green functions are affected by infra-red divergences; an adiabatic limit must be performed and, as it can be seen from the paper of Epstein and Glaser, this limit is not easy to perform.

So, for the moment, it is safer to consider the causal formalism is not equivalent to the functional formalism and study gauge theories in an independent way. In particular, the problem of anomalies produced by loop contributions is very interesting. In the functional formalism it is asserted that such anomalies do exist and the basic example is the famous axial anomaly. We will prove in this paper that in the causal formalism the loop contributions, up to the third order of the perturbation theory, do not produce anomalies! The basic idea is to isolate some typical numerical distributions with causal support appearing in the loop contributions in the second and the third order of the perturbation theory; then we prove that some identities verified by these distributions can be causally split without anomalies. This idea is in the spirit of the master Ward identity considered in the literature [2], [4], but the actual proof of our identities seems to be considerably different. In the next Section we will give a minimal account of the gauge theories in the causal approach. Then in Section 3 and 4 we prove that the loop contributions do not produce anomalies in the second and respectively the third order of the perturbation theory. Finally, in Section 5 we prove that the annulment of the axial anomaly follows from the condition that the chronological products should be coboundaries up to super-renormalizable terms.

2 General Gauge Theories

We give here the essential ingredients of perturbation theory. Suppose that the Wick monomials $W_1, \dots, W_n$ are self-adjoint: $W_j^\dagger = W_j, \ \forall j = 1, \dots, n$. The chronological products $T(W_1(x_1), \dots, W_n(x_n)) \quad n = 1, 2, \dots$ are verifying the following set of axioms:

- Skew-symmetry in all arguments $W_1(x_1), \dots, W_n(x_n)$:

$$T(\dots, W_i(x_i), W_{i+1}(x_{i+1}), \dots,) = (-1)^{f_i f_{i+1}} T(\dots, W_{i+1}(x_{i+1}), W_i(x_i), \dots) \tag{2.1}$$

  where f_i is the number of Fermi fields appearing in the Wick monomial W_i.

- Poincaré invariance: we have a natural action of the Poincaré group in the space of Wick monomials and we impose that for all $(a, A) \in inSL(2, \mathbb{C})$ we have:

$$U_{a,A} T(W_1(x_1), \dots, W_n(x_n)) U_{a,A}^{-1} = T(A \cdot W_1(A \cdot x_1 + a), \dots, A \cdot W_n(A \cdot x_n + a)); \tag{2.2}$$

  Sometimes it is possible to supplement this axiom by other invariance properties: space and/or time inversion, charge conjugation invariance, global symmetry invariance with respect to some internal symmetry group, supersymmetry, etc.

- Causality: if $x_i \geq x_j, \quad \forall i \leq k, \quad j \geq k+1$ then we have:

$$T(W_1(x_1), \dots, W_n(x_n)) = T(W_1(x_1), \dots, W_k(x_k)) \ \ T(W_{k+1}(x_{k+1}), \dots, W_n(x_n)); \tag{2.3}$$

- Unitarity: We define the *anti-chronological products* according to

$$(-1)^n \bar{T}(W_1(x_1), \dots, W_n(x_n)) \equiv \sum_{r=1}^{n} (-1)^r \sum_{I_1, \dots, I_r \in Part(\{1, \dots, n\})} \epsilon \ \ T_{I_1}(X_1) \cdots T_{I_r}(X_r) \tag{2.4}$$

  where the we have used the notation:

$$T_{\{i_1, \dots, i_k\}}(x_{i_1}, \dots, x_{i_k}) \equiv T(W_{i_1}(x_{i_1}), \dots, W_{i_k}(x_{i_k})) \tag{2.5}$$

  and the sign ϵ counts the permutations of the Fermi factors. Then the unitarity axiom is:

$$\bar{T}(W_1(x_1), \dots, W_n(x_n)) = T(W_1(x_1), \dots, W_n(x_n))^\dagger. \tag{2.6}$$

- The "initial condition"

$$T(W(x)) = W(x). \tag{2.7}$$

It can be proved that this system of axioms can be supplemented with

$$\begin{aligned} & T(W_1(x_1), \dots, W_n(x_n)) \\ = \sum \quad < \Omega, T(W_1'(x_1), \dots, W_n'(x_n)) \Omega > \quad & : W_1''(x_1), \dots, W_n''(x_n) : \end{aligned} \tag{2.8}$$

where W_i' and W_i'' are Wick submonomials of W_i such that $W_i =: W_i' W_i'' :$ and appropriate signs should be included if Fermi fields are present; here Ω is the vacuum state. This is called the *Wick expansion property.*

We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated to arbitrary Wick monomials $W_1, \dots, W_n$; explicitly:

$$\omega(< \Omega, T^{W_1,\dots,W_n}(X)\Omega >) \leq \sum_{l=1}^{n} \omega(W_l) - 4(n-1) \tag{2.9}$$

where by $\omega(d)$ we mean the order of singularity of the (numerical) distribution d and by $\omega(W)$ we mean the canonical dimension of the Wick monomial W; in particular this means that we have

$$T(W_1(x_1), \dots, W_n(x_n)) = \sum_{g} t_g(x_1, \dots, x_n)\ W_g(x_1, \dots, x_n) \tag{2.10}$$

where W_g are Wick polynomials of fixed canonical dimension and t_g are distributions in $n-1$ variables (because of translation invariance) with the order of singularity bounded by the power counting theorem [5]:

$$\omega(t_g) + \omega(W_g) \leq \sum_{j=1}^{n} \omega(W_j) - 4(n-1) \tag{2.11}$$

and the sum over g is essentially a sum over Feynman graphs. The contributions verifying the strict inequality above i.e. with the strict inequality $<$ sign, will be called *super-renormalizable* as in [8].

Up to now, we have defined the chronological products only for self-adjoint Wick monomials $W_1, \dots, W_n$ but we can extend the definition for arbitrary Wick polynomials by linearity.

One can modify the chronological products without destroying the basic property of causality *iff* one can make

$$t_g \to t_g + P_g(\partial)\delta(x_1 - x_n) \cdots \delta(x_{n-1} - x_n) \tag{2.12}$$

with P_g a monomials in the partial derivatives. If we want to preserve (2.11) we impose the restriction

$$deg(P_g) + \omega(W_g) \leq \sum_{j=1}^{n} \omega(W_j) - 4(n-1) \tag{2.13}$$

and some other restrictions are following from Lorentz covariance and unitarity.

From now on we consider that we work in the four-dimensional Minkowski space and we have the Wick polynomials T^I such that the descent equations (1.7) are true and we also have

$$T^I(x_1)\ T^J(x_2) = (-1)^{|I||J|}\ T^J(x_2)\ T^I(x_1), \quad \forall\ x_1 \sim x_2 \tag{2.14}$$

i.e. for $x_1 - x_2$ space-like these expressions causally commute in the graded sense. The chronological products $T^{I_1,\dots,I_n}(x_1, \dots, x_n)$ are constructed according to the prescription (1.8) from the Introduction and they form a cohomological object.

3 Second Order Causal Distributions

In second and third order we have some typical distributions. We remind the fact that the Pauli-Villars distribution is defined by

$$D_m(x) = D_m^{(+)}(x) + D_m^{(-)}(x) \tag{3.1}$$

where

$$D_m^{(\pm)}(x) \sim \int dp e^{ip\cdot x} \theta(\pm p_0)\delta(p^2 - m^2) \tag{3.2}$$

such that

$$D^{(-)}(x) = -D^{(+)}(-x). \tag{3.3}$$

This distribution has causal support. In fact, it can be causally split (uniquely) into an advanced and a retarded part:

$$D = D^{\rm adv} - D^{\rm ret} \tag{3.4}$$

and then we can define the Feynman propagator and antipropagator

$$D^F = D^{\rm ret} + D^{(+)}, \qquad \bar{D}^F = D^{(+)} - D^{\rm adv}. \tag{3.5}$$

All these distributions have singularity order $\omega(D) = -2$.

For one-loop contributions in the second order we need the basic distributions

$$d^{(2)}_{D_1,D_2}(x) \equiv \frac{1}{2}[D_1^{(+)}(x)\ D_2^{(+)}(x) - D_1^{(+)}(-x)\ D_2^{(+)}(x)] \tag{3.6}$$

where $D_j = D_{m_j}$ which also with causal support. This expression is linear in D_1 and D_2. We will also use the notation

$$d_{12} = d^{(2)}_{D_1,D_2} \tag{3.7}$$

and when no confusion about the distributions $D_j = D_{m_j}$ can appear, we skip all indexes altogether. The causal split

$$d_{12} = d_{12}^{adv} - d_{12}^{ret} \tag{3.8}$$

is not unique because $\omega(d_{12}) = 0$ so we make the redefinitions

$$d_{12}^{adv(ret)}(x) \rightarrow d_{12}^{adv(ret)}(x) + c\ \delta(x) \tag{3.9}$$

without affecting the support properties and the order of singularity. The corresponding Feynman propagators can be defined as above.

We have associated distributions

$$\mathcal{D}_1^\mu d_{D_1,D_2} \equiv d_{\partial^\mu D_1,D_2}, \qquad \mathcal{D}_2^\mu d_{D_1,D_2} \equiv d_{D_1,\partial^\mu D_2}, \tag{3.10}$$

etc. The operators $\mathcal{D}_j^\mu$ are not true derivatives: they commute but they do not satisfy Leibniz rule. We note that

$$\frac{\partial}{\partial x_\mu} d = \mathcal{D}_1^\mu d + \mathcal{D}_2^\mu d \tag{3.11}$$

and we ask if we can make the distribution splitting in such a way that we preserve this identity. Indeed we have an elementary result:

Theorem 3.1 *The causal splitting can be done in such a way that the previous identity is preserved i.e.*

$$\frac{\partial}{\partial x_\mu} d^{adv(ret)} = (\mathcal{D}_1^\mu d)^{adv(ret)} + (\mathcal{D}_2^\mu d)^{adv(ret)} \tag{3.12}$$

so

$$\frac{\partial}{\partial x_\mu} d^F = (\mathcal{D}_1^\mu d)^F + (\mathcal{D}_2^\mu d)^F \tag{3.13}$$

Proof: The usual argument of the causal splitting gives us

$$\frac{\partial}{\partial x_\mu} d^{adv(ret)} = (\mathcal{D}_1^\mu d)^{adv(ret)} + (\mathcal{D}_2^\mu d)^{adv(ret)} + P^\mu(\partial)\delta(x-y) \tag{3.14}$$

where P^μ are polynomials in the partial derivatives of maximal degree $deg(P^\mu) \leq 1$. Now we make the redefinitions

$$(\mathcal{D}_j^\mu d)^{adv(ret)} \to (\mathcal{D}_j^\mu d)^{adv(ret)} + Q_j^\mu(\partial)\delta \tag{3.15}$$

with Q_j^μ polynomials in the partial derivatives of maximal degree $deg(Q_j^\mu) \leq 1$ without affecting the order of singularity of the distributions and the support properties. In this way we produce a redefinition of the polynomials P^μ

$$P^\mu \to P^\mu + Q_1^\mu + Q_2^\mu \tag{3.16}$$

so if we take convenient polynomials Q_j^μ we can make $P^\mu = 0$. ■

Now we go further and observe that we have

$$\frac{\partial}{\partial x_\mu} \mathcal{D}_j^\nu d = \mathcal{D}_1^\mu \mathcal{D}_j^\nu d + \mathcal{D}_2^\mu \mathcal{D}_j^\nu d \tag{3.17}$$

and we also can split causally this relation. The proof is less elementary.

Theorem 3.2 *The causal splitting can be done in such a way that the previous identity is preserved i.e.*

$$\frac{\partial}{\partial x_\mu} (\mathcal{D}_j^\nu d)^{adv(ret)} = (\mathcal{D}_1^\mu \mathcal{D}_j^\nu d)^{adv(ret)} + (\mathcal{D}_2^\mu \mathcal{D}_j^\nu d)^{adv(ret)} \tag{3.18}$$

so

$$\frac{\partial}{\partial x_\mu} (\mathcal{D}_j^\nu d)^F = (\mathcal{D}_1^\mu \mathcal{D}_j^\nu d)^F + (\mathcal{D}_2^\mu \mathcal{D}_j^\nu d)^F. \tag{3.19}$$

Proof: As above, the causal splitting gives

$$\frac{\partial}{\partial x_\mu} (\mathcal{D}_j^\nu d)^{adv(ret)} = (\mathcal{D}_1^\mu \mathcal{D}_j^\nu d)^{adv(ret)} + (\mathcal{D}_2^\mu \mathcal{D}_j^\nu d)^{adv(ret)} + P_j^{\mu\nu}(\partial)\delta \tag{3.20}$$

with $P_j^{\mu\nu}$ polynomials in the partial derivatives of maximal degree $deg(P_j^{\mu\nu}) \leq 2$ and Lorentz covariant. We make the redefinition

$$(\mathcal{D}_j^\mu \mathcal{D}_k^\nu d)^{adv(ret)} \to (\mathcal{D}_j^\mu \mathcal{D}_k^\nu d)^{adv(ret)} + Q_{jk}^{\mu\nu}(\partial)\delta \tag{3.21}$$

with $Q_{jk}^{\mu\nu}$ polynomials in the partial derivatives of degree $deg(Q_{jk}^{\mu\nu}) \leq 2$ and Lorentz covariant; we do not affect the order of singularity and the support properties. It is clear that we must have the symmetry property:

$$Q_{jk}^{\mu\nu} = Q_{kj}^{\nu\mu}. \tag{3.22}$$

This redefinitions induces a redefinition of the "anomaly" P namely:

$$P_j^{\mu\nu} \rightarrow P_j^{\mu\nu} + Q_{1j}^{\mu\nu} + Q_{2j}^{\mu\nu}. \tag{3.23}$$

In this case we need the generic forms

$$P_j^{\mu\nu}(\partial) = a_j \partial^\mu \partial^\nu + b_j \eta^{\mu\nu} \Box + c_j \eta^{\mu\nu} \tag{3.24}$$

and

$$Q_{jk}^{\mu\nu}(\partial) = A_{jk} \partial^\mu \partial^\nu + B_{jk} \eta^{\mu\nu} \Box + C_{jk} \eta^{\mu\nu} \tag{3.25}$$

where the symmetry property imposes

$$A_{jk} = A_{kj}, \quad B_{jk} = B_{kj}, \quad C_{jk} = C_{kj}. \tag{3.26}$$

The redefinition above is then

$$a_j \rightarrow a_j + A_{1j} + A_{2j} \quad b_j \rightarrow b_j + B_{1j} + B_{2j} \quad c_j \rightarrow c_j + C_{1j} + C_{2j} \tag{3.27}$$

and it is easy to see that if we take convenient matrices A, B, C we can make the null vectors a, b, c i.e. we eliminate the "anomaly" P. ■

Finally we have the identity

$$\frac{\partial}{\partial x_\mu} \mathcal{D}_j^\nu \mathcal{D}_k^\rho d = \mathcal{D}_1^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho d + \mathcal{D}_2^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho d \tag{3.28}$$

and we also can split causally this relation.

Theorem 3.3 *The causal splitting can be done in such a way that the previous identity is preserved i.e.*

$$\frac{\partial}{\partial x_\mu} (\mathcal{D}_j^\nu \mathcal{D}_k^\rho d)^{adv(ret)} = (\mathcal{D}_1^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho d)^{adv(ret)} + (\mathcal{D}_2^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho d)^{adv(ret)} \tag{3.29}$$

so we also have

$$\frac{\partial}{\partial x_\mu} (\mathcal{D}_j^\nu \mathcal{D}_k^\rho d)^F = (\mathcal{D}_1^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho d)^F + (\mathcal{D}_2^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho d)^F \tag{3.30}$$

Proof: As above we have after causal splitting

$$\frac{\partial}{\partial x_\mu} (\mathcal{D}_j^\nu \mathcal{D}_k^\rho d)^{adv(ret)} = (\mathcal{D}_1^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho d)^{adv(ret)} + (\mathcal{D}_2^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho d)^{adv(ret)} + P_{jk}^{\mu\nu\rho}(\partial)\delta \tag{3.31}$$

where $P_{jk}^{\mu\nu\rho}$ polynomials in the partial derivatives of maximal degree $deg(P_{jk}^{\mu\nu\rho}) \leq 3$, Lorentz covariant and with the symmetry property

$$P_{jk}^{\mu\nu\rho} = P_{kj}^{\mu\rho\nu}. \tag{3.32}$$

We make the redefinition

$$(\mathcal{D}_j^\mu \mathcal{D}_k^\nu \mathcal{D}_l^\rho d)^{adv(ret)} \rightarrow (\mathcal{D}_j^\mu \mathcal{D}_k^\nu \mathcal{D}_l^\rho d)^{adv(ret)} + Q_{jkl}^{\mu\nu\rho}(\partial)\delta \tag{3.33}$$

with $Q_{jkl}^{\mu\nu\rho}$ polynomials in the partial derivatives of degree $deg(Q_{jkl}^{\mu\nu\rho}) \leq 3$, Lorentz covariant and with the symmetry property:

$$Q_{jkl}^{\mu\nu\rho} = Q_{kjl}^{\nu\mu\rho} = Q_{jlk}^{\mu\rho\nu}. \tag{3.34}$$

We do not affect the order of singularity and the support properties. This redefinitions induces a redefinition of the "anomaly" P namely:

$$P_{jk}^{\mu\nu\rho} \rightarrow P_{jk}^{\mu\nu\rho} + Q_{1jk}^{\mu\nu\rho} + Q_{2jk}^{\mu\nu\rho}. \tag{3.35}$$

However in this case the redefinition seems not to work i.e. we cannot put to zero the anomaly. Indeed if we write the generic form

$$P_{jk}^{\mu\nu\rho}(\partial) = a_{jk}\ \partial^\mu\partial^\nu\partial^\rho + b_{jk}\ \eta^{\mu\nu}\partial^\rho\Box + b_{kj}\ \eta^{\mu\rho}\partial^\nu\Box + c_{jk}\ \eta^{\nu\rho}\partial^\mu\Box + d_{jk}\ \eta^{\mu\nu}\partial^\rho + d_{kj}\ \eta^{\mu\rho}\partial^\nu + e_{jk}\ \eta^{\nu\rho}\partial^\mu \tag{3.36}$$

with the matrices a, c, e symmetric, and a similar form for $Q_{jkl}^{\mu\nu\rho}$ then by the process of redefinition, we can make null all the expressions except $c_{11}, c_{22}, e_{11}, e_{22}$.

The key is to consider the expression

$$\frac{\partial^2}{\partial x_\mu \partial x_\nu}(\mathcal{D}_k^\rho d)^{adv(ret)} \tag{3.37}$$

which is obvious symmetric in $\mu \leftrightarrow \nu$ and express it using the preceding proposition and formula (3.31):

$$\begin{aligned}\frac{\partial^2}{\partial x_\mu \partial x_\nu}(\mathcal{D}_k^\rho d)^{adv(ret)} &= (\mathcal{D}_1^\mu \mathcal{D}_1^\nu \mathcal{D}_k^\rho d)^{adv(ret)} + (\mathcal{D}_2^\mu \mathcal{D}_1^\nu \mathcal{D}_k^\rho d)^{adv(ret)} + P_{1k}^{\mu\nu\rho}(\partial)\delta \\ &+ (\mathcal{D}_2^\mu \mathcal{D}_2^\nu \mathcal{D}_k^\rho d)^{adv(ret)} + (\mathcal{D}_1^\mu \mathcal{D}_2^\nu \mathcal{D}_k^\rho d)^{adv(ret)} + P_{2k}^{\mu\nu\rho}(\partial)\delta\end{aligned} \tag{3.38}$$

In this way we obtain the restriction

$$P_{1k}^{\mu\nu\rho} + P_{2k}^{\mu\nu\rho} = \mu \leftrightarrow \nu. \tag{3.39}$$

If we impose this symmetry property we find out that in fact $c_{11} = c_{22} = e_{11} = e_{22} = 0$ so we have succeeded to make $P_{jk}^{\mu\nu\rho} = 0$ and this finishes the proof. ■

Now we can use the preceding results to prove:

Theorem 3.4 *In the Yang-Mills case the one-loop contributions cannot produce anomalies in the second order perturbation theory.*

Proof: The one-loop contribution to the causal commutators $D^{I,J}(x,y) = [T^I(x), T^J(y)]$ are sums of contributions of the type

$$d^{(2)}_{D_1,D_2}\, W(x,y), \quad \mathcal{D}^\mu_j d^{(2)}_{D_1,D_2}\, W(x,y), \quad \mathcal{D}^\mu_j \mathcal{D}^\nu_k d^{(2)}_{D_1,D_2}\, W(x,y) \tag{3.40}$$

where W are Wick monomials. These expressions come with various numerical coefficients t (which are in fact Lorentz tensors). This follows from the limitations of the Yang-Mills model: we have at most a derivative in the interaction Lagrangian, so we have at most two derivatives $\mathcal{D}$ on d. The gauge invariance (1.7) gives immediately

$$d_Q D^{I,J} = i \frac{\partial}{\partial x^\mu} D^{I\mu,J} + (-1)^{|I|} \frac{\partial}{\partial y^\mu} D^{I,J\mu} \tag{3.41}$$

and in particular for the one-loop contribution

$$d_Q D^{I,J}_{(1)} = i \frac{\partial}{\partial x^\mu} D^{I\mu,J}_{(1)} + (-1)^{|I|} \frac{\partial}{\partial y^\mu} D^{I,J\mu}_{(1)}. \tag{3.42}$$

If we substitute in this relation the expression of the one-loop contributions as a sum of expressions of the type (3.40) with various numerical coefficients, then we will obtain some identities between the coefficients t **modulo** the identities (3.11), (3.17), (3.28). These identities are worked out in detail in the literature ([10] and refs. cited there). So if we can causally split these relations without anomalies we will obtain from the preceding relation

$$\begin{aligned} d_Q A^{I,J}_{(1)} &= i \frac{\partial}{\partial x^\mu} A^{I\mu,J}_{(1)} + (-1)^{|I|} \frac{\partial}{\partial y^\mu} A^{I,J\mu}_{(1)} \\ d_Q R^{I,J}_{(1)} &= i \frac{\partial}{\partial x^\mu} R^{I\mu,J}_{(1)} + (-1)^{|I|} \frac{\partial}{\partial y^\mu} R^{I,J\mu}_{(1)} \end{aligned} \tag{3.43}$$

so finally

$$d_Q T^{I,J}_{(1)} = i \frac{\partial}{\partial x^\mu} T^{I\mu,J}_{(1)} + (-1)^{|I|} \frac{\partial}{\partial y^\mu} T^{I,J\mu}_{(1)} \tag{3.44}$$

i.e. gauge invariance for the one-loop contributions. But we just have proved in the preceding three theorems that such a causal splitting can be done. ■

Remark 3.5 *If we consider gravity, then we must extend the list (3.40) up to four derivatives $\mathcal{D}$ and we must investigate accordingly if the causal splitting is possible without anomalies.*

We can reduce the two loop contributions case to the one-loop case if we observe that the corresponding distributions can be obtained if in $d_{12} = d_{D_1,D_2}$ we replace $D_2 \to d_{23} = d_{D_2,D_2}$.

4 Third Order Causal Distributions

For the triangle one-loop contributions in the third order we need the relevant distributions with causal support. First, we take $D_j = D_{m_j}, j = 1,2,3$ and define

$$\begin{aligned} d^{(3)}_{D_1,D_2,D_3}(x,y,z) \equiv \bar{D}^F_3(x-y)[D^{(-)}_2(z-x)D^{(+)}_1(y-z) - D^{(+)}_2(z-x)D^{(-)}_1(y-z)] \\ +D^F_1(y-z)[D^{(-)}_3(x-y)D^{(+)}_2(z-x) - D^{(+)}_3(x-y)D^{(-)}_2(z-x)] \\ +D^F_2(z-x)[D^{(-)}_1(y-z)D^{(+)}_3(x-y) - D^{(+)}_1(y-z)D^{(-)}_3(x-y)] \end{aligned} \tag{4.1}$$

which also with causal support; indeed we have the alternative forms

$$\begin{aligned} d^{(3)}_{D_1,D_2,D_3}(x,y,z) = -D^{\rm ret}_3(x-y)[D^{(-)}_2(z-x)D^{(+)}_1(y-z) - D^{(+)}_2(z-x)D^{(-)}_1(y-z)] \\ +D^{\rm adv}_1(y-z)[D^{(-)}_3(x-y)D^{(+)}_2(z-x) - D^{(+)}_3(x-y)D^{(-)}_2(z-x)] \\ +D^{\rm adv}_2(z-x)[D^{(-)}_1(y-z)D^{(+)}_3(x-y) - D^{(+)}_1(y-z)D^{(-)}_3(x-y)] \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} d^{(3)}_{D_1,D_2,D_3}(x,y,z) = -D^{\rm adv}_3(x-y)[D^{(-)}_2(z-x)D^{(+)}_1(y-z) - D^{(+)}_2(z-x)D^{(-)}_1(y-z)] \\ +D^{\rm ret}_1(y-z)[D^{(-)}_3(x-y)D^{(+)}_2(z-x) - D^{(+)}_3(x-y)D^{(-)}_2(z-x)] \\ +D^{\rm ret}_2(z-x)[D^{(-)}_1(y-z)D^{(+)}_3(x-y) - D^{(+)}_1(y-z)D^{(-)}_3(x-y)] \end{aligned} \tag{4.3}$$

from which it follows that $d^{(3)}_{D_1,D_2,D_3}(x,y,z)$ is null outside the causal cone $\{(x,y,z)|x-z \in V^+, y-z \in V^+\} \cup \{(x,y,z)|x-z \in V^-, y-z \in V^-\}$. These distributions have the singularity order $\omega(d^{(3)}_{D_1,D_2,D_3}) = -2$.

As in the previous Section we use the alternative notation

$$d_{123} \equiv d^{(3)}_{D_1,D_2,D_3} \tag{4.4}$$

and when there is no ambiguity about the distributions D_j we simply denote $d = d_{123}$. There are some associated distributions obtained from $d_{D_1,D_2,D_3}(x,y,z)$ applying derivatives on the factors $D_j = D_{m_j}, j = 1,2,3$. For instance we denote

$$\begin{aligned} \mathcal{D}^1_\alpha d_{D_1,D_2,D_3} \equiv d_{\partial_\alpha D_1,D_2,D_3} \\ \mathcal{D}^2_\alpha d_{D_1,D_2,D_3} \equiv d_{D_1,\partial_\alpha D_2,D_3} \\ \mathcal{D}^3_\alpha d_{D_1,D_2,D_3} \equiv d_{D_1,D_2,\partial_\alpha D_3}, \end{aligned} \tag{4.5}$$

and so on for more derivatives ∂_α distributed in an arbitrary way on the factors $D_j = D_{m_j}, j = 1,2,3$. We mention the fact that the operators $\mathcal{D}^j_\alpha, j = 1,2,3$ are commutative but they are not derivation operators: they do not verify Leibniz rule. It will be convenient to denote

$$\mathcal{K}_j = \mathcal{D}^\alpha_j \mathcal{D}_{j\alpha},\ \ j = 1,2,3. \tag{4.6}$$

We note the formulas

$$\frac{\partial}{\partial x_\mu} d = (\mathcal{D}^\mu_3 - \mathcal{D}^\mu_2)d, \quad \frac{\partial}{\partial y_\mu} d = (\mathcal{D}^\mu_1 - \mathcal{D}^\mu_3)d, \quad \frac{\partial}{\partial z_\mu} d = (\mathcal{D}^\mu_2 - \mathcal{D}^\mu_1)d. \tag{4.7}$$

As in the preceding Section we investigate if such formulas can be causally split. For the preceding formulas this is trivial

Theorem 4.1 *The preceding formulas can be causally split without anomalies i.e.*

$$\begin{aligned}
\frac{\partial}{\partial x_\mu} d^{adv(ret)} &= (\mathcal{D}_3^\mu d)^{adv(ret)} - (\mathcal{D}_2^\mu d)^{adv(ret)} \\
\frac{\partial}{\partial y_\mu} d^{adv(ret)} &= (\mathcal{D}_1^\mu d)^{adv(ret)} - (\mathcal{D}_3^\mu d)^{adv(ret)} \\
\frac{\partial}{\partial z_\mu} d^{adv(ret)} &= (\mathcal{D}_2^\mu d)^{adv(ret)} - (\mathcal{D}_1^\mu d)^{adv(ret)}.
\end{aligned} \tag{4.8}$$

Proof: By causal splitting, we can obtain an anomaly of the type $P(\partial)\delta(x,y,z)$ with P a polynomial in the partial derivatives and

$$\delta(x,y,z) = \delta(x-y)\ \delta(y-z)\ \delta(z-x) \tag{4.9}$$

and such an anomaly has the order of singularity equal to the degree of the polynomial P. But both sides of the formulas (4.7) have order of singularity $\omega = -1$ so the anomaly must be null.
■

As in the preceding Section, we go further and consider the identities

$$\begin{aligned}
\frac{\partial}{\partial x_\mu} \mathcal{D}_j^\nu d &= \mathcal{D}_3^\mu \mathcal{D}_j^\nu d - \mathcal{D}_2^\mu \mathcal{D}_j^\nu d \\
\frac{\partial}{\partial y_\mu} \mathcal{D}_j^\nu d &= \mathcal{D}_1^\mu \mathcal{D}_j^\nu d - \mathcal{D}_3^\mu \mathcal{D}_j^\nu d \\
\frac{\partial}{\partial z_\mu} \mathcal{D}_j^\nu d &= \mathcal{D}_2^\mu \mathcal{D}_j^\nu d - \mathcal{D}_1^\mu \mathcal{D}_j^\nu d
\end{aligned} \tag{4.10}$$

and prove

Theorem 4.2 *The preceding formulas can be causally split without anomalies i.e.*

$$\begin{aligned}
\frac{\partial}{\partial x_\mu} (\mathcal{D}_j^\nu d)^{adv(ret)} &= (\mathcal{D}_3^\mu \mathcal{D}_j^\nu d)^{adv(ret)} - (\mathcal{D}_2^\mu \mathcal{D}_j^\nu d)^{adv(ret)} \\
\frac{\partial}{\partial y_\mu} (\mathcal{D}_j^\nu d)^{adv(ret)} &= (\mathcal{D}_1^\mu \mathcal{D}_j^\nu d)^{adv(ret)} - (\mathcal{D}_3^\mu \mathcal{D}_j^\nu d)^{adv(ret)} \\
\frac{\partial}{\partial z_\mu} (\mathcal{D}_j^\nu d)^{adv(ret)} &= (\mathcal{D}_2^\mu \mathcal{D}_j^\nu d)^{adv(ret)} - (\mathcal{D}_1^\mu \mathcal{D}_j^\nu d)^{adv(ret)}.
\end{aligned} \tag{4.11}$$

These relations remain invariant with respect to the redefinition

$$(\mathcal{D}_j^\mu \mathcal{D}_k^\nu d)^{adv(ret)} \rightarrow (\mathcal{D}_j^\mu \mathcal{D}_k^\nu d)^{adv(ret)} + b\ \eta^{\mu\nu}\ \delta. \tag{4.12}$$

Proof: The causal splitting applied to the relations (4.10) leads to

$$\frac{\partial}{\partial x_\mu}(\mathcal{D}^\nu_j d)^{adv(ret)} = (\mathcal{D}^\mu_3 \mathcal{D}^\nu_j d)^{adv(ret)} - (\mathcal{D}^\mu_2 \mathcal{D}^\nu_j d)^{adv(ret)} + c_{1j}\ \eta^{\mu\nu}\ \delta$$
$$\frac{\partial}{\partial y_\mu}(\mathcal{D}^\nu_j d)^{adv(ret)} = (\mathcal{D}^\mu_1 \mathcal{D}^\nu_j d)^{adv(ret)} - (\mathcal{D}^\mu_3 \mathcal{D}^\nu_j d)^{adv(ret)} + c_{2j}\ \eta^{\mu\nu}\ \delta$$
$$\frac{\partial}{\partial z_\mu}(\mathcal{D}^\nu_j d)^{adv(ret)} = (\mathcal{D}^\mu_2 \mathcal{D}^\nu_j d)^{adv(ret)} - (\mathcal{D}^\mu_1 \mathcal{D}^\nu_j d)^{adv(ret)} + c_{3j}\ \eta^{\mu\nu}\ \delta \tag{4.13}$$

where c_{jk} is a constant matrix. Now we consider the expressions

$$\frac{\partial^2}{\partial x_\mu \partial y_\nu} d^{adv(ret)}, \quad \frac{\partial^2}{\partial y_\mu \partial z_\nu} d^{adv(ret)}, \quad \frac{\partial^2}{\partial x_\mu \partial z_\nu} d^{adv(ret)} \tag{4.14}$$

and write them in two possible ways using the preceding formulas and the preceding theorem. We obtain the consistency relations:

$$c_{11} - c_{13} = c_{23} - c_{22}$$
$$c_{22} - c_{21} = c_{31} - c_{33}$$
$$c_{33} - c_{32} = c_{12} - c_{11}. \tag{4.15}$$

From here we can express the diagonal elements of the matrix c_{jk} in terms of the non-diagonal elements:

$$c_{11} = \frac{1}{2}(c_{12} + c_{13} + c_{23} + c_{32} - c_{21} - c_{31}) \tag{4.16}$$

and two more by circular permutations. Now we make the redefinitions

$$(\mathcal{D}^\mu_j \mathcal{D}^\nu_k d)^{adv(ret)} \rightarrow (\mathcal{D}^\mu_j \mathcal{D}^\nu_k d)^{adv(ret)} + b_{jk}\ \eta^{\mu\nu}\ \delta \tag{4.17}$$

with b_{jk} a symmetric matrix without affection the support properties and the order of singularity. As a result we have

$$c_{1j} \rightarrow c_{1j} - b_{3j} + b_{2j}$$
$$c_{2j} \rightarrow c_{2j} - b_{1j} + b_{3j}$$
$$c_{3j} \rightarrow c_{3j} - b_{2j} + b_{3j}. \tag{4.18}$$

If we take convenient b_{jk} we can make null the non-diagonal elements of the matrix c_{jk} and then if we use the consistency relations we get that the diagonal elements c_{jj} are null too. ■

The next step is considerably more complicated. We start from the identities

$$\frac{\partial}{\partial x_\mu}\mathcal{D}^\nu_j \mathcal{D}^\rho_k d = \mathcal{D}^\mu_3 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d - \mathcal{D}^\mu_2 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d$$
$$\frac{\partial}{\partial y_\mu}\mathcal{D}^\nu_j \mathcal{D}^\rho_k d = \mathcal{D}^\mu_1 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d - \mathcal{D}^\mu_3 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d$$
$$\frac{\partial}{\partial z_\mu}\mathcal{D}^\nu_j \mathcal{D}^\rho_k d = \mathcal{D}^\mu_2 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d - \mathcal{D}^\mu_1 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d \tag{4.19}$$

and prove

Theorem 4.3 *The preceding relations can be causally split in such a way that we have*

$$\frac{\partial}{\partial x_\mu}(\mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} = (\mathcal{D}^\mu_3 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} - (\mathcal{D}^\mu_2 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} + \mathcal{A}^{\mu\nu\rho}_{1jk}$$
$$\frac{\partial}{\partial y_\mu}(\mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} = (\mathcal{D}^\mu_1 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} - (\mathcal{D}^\mu_3 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} + \mathcal{A}^{\mu\nu\rho}_{2jk}$$
$$\frac{\partial}{\partial z_\mu}(\mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} = (\mathcal{D}^\mu_2 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} - (\mathcal{D}^\mu_1 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} + \mathcal{A}^{\mu\nu\rho}_{3jk} \tag{4.20}$$

where the anomalies have the structure

$$\mathcal{A}^{\mu\nu\rho}_{jkl} = \sum_{p=1}^{2} a^p_{jkl}(\eta^{\mu\nu}\ \partial^\rho_p + \eta^{\mu\rho}\ \partial^\nu_p + \eta^{\nu\rho}\ \partial^\mu_p)\ \delta \tag{4.21}$$

with the constants a^p_{jkl} symmetric in $k \leftrightarrow l$. Moreover, we can take non-null only the constants

$$a^p_{111} = a^p_{222} = a^p_{333} = a^p_{123} = a^p_{213} = a^p_{312} = a^p. \tag{4.22}$$

The preceding relations are invariant to the redefinitions

$$(\mathcal{D}^\mu_j \mathcal{D}^\nu_k \mathcal{D}^\rho_l d)^{adv(ret)} \rightarrow (\mathcal{D}^\mu_j \mathcal{D}^\nu_k \mathcal{D}^\rho_l d)^{adv(ret)} + \sum_{p=1}^{2} A^p(\eta^{\mu\nu}\ \partial^\rho_p + \eta^{\mu\rho}\ \partial^\nu_p + \eta^{\nu\rho}\ \partial^\mu_p)\ \delta. \tag{4.23}$$

Proof: The causal splitting of the relations (4.19) leads to

$$\frac{\partial}{\partial x_\mu}(\mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} = (\mathcal{D}^\mu_3 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} - (\mathcal{D}^\mu_2 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} + P^{\mu\nu\rho}_{1jk}(\partial)\delta$$
$$\frac{\partial}{\partial y_\mu}(\mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} = (\mathcal{D}^\mu_1 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} - (\mathcal{D}^\mu_3 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} + P^{\mu\nu\rho}_{2jk}(\partial)\delta$$
$$\frac{\partial}{\partial z_\mu}(\mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} = (\mathcal{D}^\mu_2 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} - (\mathcal{D}^\mu_1 \mathcal{D}^\nu_j \mathcal{D}^\rho_k d)^{adv(ret)} + P^{\mu\nu\rho}_{3jk}(\partial)\delta \tag{4.24}$$

where $P^{\mu\nu\rho}_{ijk}$ are polynomials in the partial derivatives of degree $\deg(P^{\mu\nu\rho}_{ijk}) \leq 1$, Lorentz covariant and with the symmetry property

$$P^{\mu\nu\rho}_{ijk} = P^{\mu\rho\nu}_{ikj}. \tag{4.25}$$

It is clear that we must consider only the partial derivatives ∂^μ_1 and ∂^μ_2 because of the identity

$$(\partial^\mu_1 + \partial^\mu_2 + \partial^\mu_3)\ \delta = 0. \tag{4.26}$$

As before, we consider the expressions

$$\frac{\partial^2}{\partial x_\mu \partial y_\nu}(\mathcal{D}^\rho_k d)^{adv(ret)}, \quad \frac{\partial^2}{\partial y_\mu \partial z_\nu}(\mathcal{D}^\rho_k d)^{adv(ret)}, \quad \frac{\partial^2}{\partial x_\mu \partial z_\nu}(\mathcal{D}^\rho_k d)^{adv(ret)} \tag{4.27}$$

and write them in two possible ways using the preceding formulas and the preceding theorem. We obtain the consistency relations:

$$
\begin{aligned}
P^{\mu\nu\rho}_{11k} - P^{\mu\nu\rho}_{13k} &= P^{\nu\mu\rho}_{23k} - P^{\nu\mu\rho}_{22k} \\
P^{\mu\nu\rho}_{22k} - P^{\mu\nu\rho}_{21k} &= P^{\nu\mu\rho}_{31k} - P^{\nu\mu\rho}_{33k} \\
P^{\mu\nu\rho}_{33k} - P^{\mu\nu\rho}_{32k} &= P^{\nu\mu\rho}_{12k} - P^{\nu\mu\rho}_{11k}.
\end{aligned}
\tag{4.28}
$$

Now we consider the expressions

$$
\frac{\partial^2}{\partial x_\mu \partial x_\nu}(\mathcal{D}^\rho_k d)^{adv(ret)}, \quad \frac{\partial^2}{\partial y_\mu \partial y_\nu}(\mathcal{D}^\rho_k d)^{adv(ret)}, \quad \frac{\partial^2}{\partial z_\mu \partial z_\nu}(\mathcal{D}^\rho_k d)^{adv(ret)}
\tag{4.29}
$$

compute them using the the preceding formulas and the preceding theorem and use the symmetry in $\mu \leftrightarrow \nu$. We obtain new consistency relations:

$$
\begin{aligned}
P^{\mu\nu\rho}_{13k} - P^{\mu\nu\rho}_{12k} &= \mu \leftrightarrow \nu \\
P^{\mu\nu\rho}_{21k} - P^{\mu\nu\rho}_{23k} &= \mu \leftrightarrow \nu \\
P^{\mu\nu\rho}_{32k} - P^{\mu\nu\rho}_{31k} &= \mu \leftrightarrow \nu.
\end{aligned}
\tag{4.30}
$$

We can make the redefinitions

$$
(\mathcal{D}^\mu_j \mathcal{D}^\nu_k \mathcal{D}^\rho_l d)^{adv(ret)} \rightarrow (\mathcal{D}^\mu_j \mathcal{D}^\nu_k \mathcal{D}^\rho_l d)^{adv(ret)} + Q^{\mu\nu\rho}_{jkl}(\partial)\ \delta
\tag{4.31}
$$

with $Q^{\mu\nu\rho}_{jkl}$ polynomials in the partial derivatives of degree $\deg(Q^{\mu\nu\rho}_{jkl}) \leq 1$, Lorentz covariant and with the symmetry property

$$
Q^{\mu\nu\rho}_{jkl} = Q^{\nu\mu\rho}_{kjl} = Q^{\mu\rho\nu}_{jlk}.
\tag{4.32}
$$

In this way we do not disturb the order of the singularity and the causal support properties. This redefinitions change the anomalies according to

$$
\begin{aligned}
P^{\mu\nu\rho}_{1jk} &\rightarrow P^{\mu\nu\rho}_{1jk} - Q^{\mu\nu\rho}_{3jk} + Q^{\mu\nu\rho}_{2jk} \\
P^{\mu\nu\rho}_{2jk} &\rightarrow P^{\mu\nu\rho}_{2jk} - Q^{\mu\nu\rho}_{1jk} + Q^{\mu\nu\rho}_{3jk} \\
P^{\mu\nu\rho}_{3jk} &\rightarrow P^{\mu\nu\rho}_{3jk} - Q^{\mu\nu\rho}_{2jk} + Q^{\mu\nu\rho}_{1jk}.
\end{aligned}
\tag{4.33}
$$

One can use these redefinitions to make null some of the the anomalies:

$$
P^{\mu\nu\rho}_{122} = P^{\mu\nu\rho}_{133} = P^{\mu\nu\rho}_{211} = P^{\mu\nu\rho}_{233} = P^{\mu\nu\rho}_{311} = P^{\mu\nu\rho}_{322} = P^{\mu\nu\rho}_{313} = 0
\tag{4.34}
$$

and we are left with some redefinition freedom

$$
\begin{aligned}
Q^{\mu\nu\rho}_{211} &= Q^{\mu\nu\rho}_{311} = Q^{\mu\nu\rho}_{111} \\
Q^{\mu\nu\rho}_{122} &= Q^{\mu\nu\rho}_{322} = Q^{\mu\nu\rho}_{222} \\
Q^{\mu\nu\rho}_{133} &= Q^{\mu\nu\rho}_{233} = Q^{\mu\nu\rho}_{333} \\
Q^{\mu\nu\rho}_{213} &= Q^{\mu\nu\rho}_{113} = Q^{\mu\nu\rho}_{111}
\end{aligned}
\tag{4.35}
$$

which we will use later. Let us consider the generic form

$$P_{123}^{\mu\nu\rho} = \sum_{p=1}^{2} (a_1^p \eta^{\mu\nu}\ \partial_p^\rho + a_2^p \eta^{\mu\rho}\ \partial_p^\nu + a_3^p \eta^{\nu\rho}\ \partial_p^\mu) \tag{4.36}$$

and use the consistency relations

$$P_{123}^{\mu\rho\nu} = \mu \leftrightarrow \nu, \qquad P_{123}^{\mu\nu\rho} = \mu \leftrightarrow \nu \tag{4.37}$$

which follows from the first relation (4.30) for $k = 2, 3$ if we take into account (4.34) also. Then we obtain that

$$a_1^p = a_2^p = a_3^p \tag{4.38}$$

so we have the generic form

$$P_{123}^{\mu\nu\rho} = \sum_{p=1}^{2} a^p\ A_p^{\mu\nu\rho} \tag{4.39}$$

where

$$A_p^{\mu\nu\rho} = \eta^{\mu\nu}\ \partial_p^\rho + \eta^{\mu\rho}\ \partial_p^\nu + \eta^{\nu\rho}\ \partial_p^\mu. \tag{4.40}$$

In the same way we obtain

$$P_{213}^{\mu\nu\rho} = \sum_{p=1}^{2} b^p\ A_p^{\mu\nu\rho} \tag{4.41}$$

and

$$P_{312}^{\mu\nu\rho} = \sum_{p=1}^{2} c^p\ A_p^{\mu\nu\rho}. \tag{4.42}$$

The remaining consistency relations can be used to determine the rest of the anomalies. We obtain

$$b^p = c^p \tag{4.43}$$

and

$$\begin{aligned}
P_{111}^{\mu\nu\rho} = P_{222}^{\mu\nu\rho} = P_{333}^{\mu\nu\rho} &= \frac{1}{3} \sum_{p=1}^{2} (a^p + 2b^p)\ A_p^{\mu\nu\rho} \\
P_{123}^{\mu\nu\rho} &= \sum_{p=1}^{2} a^p\ A_p^{\mu\nu\rho} \\
P_{213}^{\mu\nu\rho} = P_{312}^{\mu\nu\rho} &= \sum_{p=1}^{2} b^p\ A_p^{\mu\nu\rho}. \\
P_{112}^{\mu\nu\rho} = P_{113}^{\mu\nu\rho} &= \frac{1}{3} \sum_{p=1}^{2} (a^p - b^p)\ A_p^{\mu\nu\rho} \\
P_{323}^{\mu\nu\rho} = P_{223}^{\mu\nu\rho} &= -\frac{1}{3} \sum_{p=1}^{2} (a^p - b^p)\ A_p^{\mu\nu\rho}
\end{aligned} \tag{4.44}$$

Now we still can make a redefinition verifying (4.35) to make

$$P_{112}^{\mu\nu\rho} = 0 \tag{4.45}$$

and this gives

$$a^p = b^p. \tag{4.46}$$

We obtain the relations from the statement. ■

Now we show that in fact the anomaly from the preceding theorem is null.

Theorem 4.4 *The causal splitting can be done in such a way that*

$$\mathcal{A}_{jkl}^{\mu\nu\rho} = 0. \tag{4.47}$$

Proof: We consider in the first relation from the statement of the previous theorem $j = k = 1$ and contract with $\eta_{\nu\rho}$. The result is

$$\frac{\partial}{\partial x_\mu}(\mathcal{K}_1 d_{123})^{adv(ret)} = (\mathcal{D}_3^\mu \mathcal{K}_1 d)^{adv(ret)} - (\mathcal{D}_2^\mu \mathcal{K}_1 d)^{adv(ret)} + \eta_{\nu\rho}\ \mathcal{A}_{111}^{\mu\nu\rho}. \tag{4.48}$$

But it easy to prove that

$$\begin{aligned} \mathcal{K}_1 d_{123}(x,y,z) &= 2\ \delta(y-z)\ d_{23}(x-y) - m_1^2\ d_{123}(x,y,z) \\ \mathcal{D}_3^\mu \mathcal{K}_1 d_{123}(x,y,z) &= 2\ \delta(y-z)\ \mathcal{D}_3^\mu d_{23}(x-y) - m_1^2\ \mathcal{D}_3^\mu d_{123}(x,y,z) \\ \mathcal{D}_2^\mu \mathcal{K}_1 d_{123}(x,y,z) &= -2\ \delta(y-z)\ \mathcal{D}_2^\mu d_{23}(x-y) - m_1^2\ \mathcal{D}_2^\mu d_{123}(x,y,z) \end{aligned} \tag{4.49}$$

so the preceding relation becomes

$$\begin{aligned} 2\ \delta(y-z)\ \frac{\partial}{\partial x_\mu} d_{23}^{adv(ret)}(x-y) - m_1^2\ \frac{\partial}{\partial x_\mu} d_{123}^{adv(ret)}(x,y,z) = \\ 2\ \delta(y-z)\ (\mathcal{D}_3^\mu d_{23})^{adv(ret)}(x-y) - m_1^2\ (\mathcal{D}_3^\mu d_{123})^{adv(ret)}(x,y,z) \\ +2\ \delta(y-z)\ (\mathcal{D}_2^\mu d_{23})^{adv(ret)}(x-y) + m_1^2\ (\mathcal{D}_2^\mu d_{123})^{adv(ret)}(x,y,z) + \eta_{\nu\rho}\ \mathcal{A}_{111}^{\mu\nu\rho}(x,y,z). \end{aligned} \tag{4.50}$$

Now we use the preceding theorem and are left with

$$\delta(y-z)\ \left[\frac{\partial}{\partial x_\mu} d_{23}^{adv(ret)}(x-y) - (\mathcal{D}_3^\mu d_{23})^{adv(ret)}(x-y) - (\mathcal{D}_2^\mu d_{23})^{adv(ret)}(x-y)\right] = \eta_{\nu\rho}\ \mathcal{A}_{111}^{\mu\nu\rho}(x,y,z). \tag{4.51}$$

But from theorem 3.1 we have

$$\frac{\partial}{\partial x_\mu} d_{23}^{adv(ret)} = (\mathcal{D}_2^\mu d_{23})^{adv(ret)} + (\mathcal{D}_3^\mu d_{23})^{adv(ret)} \tag{4.52}$$

and so we are left with

$$\eta_{\nu\rho}\ \mathcal{A}_{111}^{\mu\nu\rho}(x,y,z) = 0. \tag{4.53}$$

If we insert the explicit form of the anomaly from the preceding theorem then we get $a^p = 0$ so

$$\mathcal{A}_{jkl}^{\mu\nu\rho} = 0. \tag{4.54}$$

■

The next (and final) step is to investigate the causal splitting of the relations

$$\begin{aligned}
\frac{\partial}{\partial x_\mu} \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d &= \mathcal{D}_3^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d - \mathcal{D}_2^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d \\
\frac{\partial}{\partial y_\mu} \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d &= \mathcal{D}_1^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d - \mathcal{D}_3^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d \\
\frac{\partial}{\partial z_\mu} \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d &= \mathcal{D}_2^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d - \mathcal{D}_1^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d.
\end{aligned} \tag{4.55}$$

The result is

Theorem 4.5 *The causal splitting can be done in such a way that*

$$\begin{aligned}
\frac{\partial}{\partial x_\mu} (\mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} &= (\mathcal{D}_3^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} - (\mathcal{D}_2^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} + P_{1jkl}^{\mu\nu\rho\sigma}(\partial)\delta \\
\frac{\partial}{\partial y_\mu} (\mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} &= (\mathcal{D}_1^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} - (\mathcal{D}_3^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} + P_{2jkl}^{\mu\nu\rho\sigma}(\partial)\delta \\
\frac{\partial}{\partial z_\mu} (\mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} &= (\mathcal{D}_2^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} - (\mathcal{D}_1^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} + P_{3jkl}^{\mu\nu\rho\sigma}(\partial)\delta
\end{aligned} \tag{4.56}$$

where we can take non-null only the anomalies

$$\begin{aligned}
P_{llll}^{\mu\nu\rho\sigma} &= P^{\mu\nu\rho\sigma}, \quad l = 1,2,3, \\
P_{1122}^{\mu\nu\rho\sigma} = P_{2233}^{\mu\nu\rho\sigma} &= P_{3311}^{\mu\nu\rho\sigma} = -P^{\mu\nu\rho\sigma}, \\
P_{2211}^{\mu\nu\rho\sigma} = P_{3322}^{\mu\nu\rho\sigma} &= P_{1133}^{\mu\nu\rho\sigma} = P^{\mu\nu\rho\sigma}, \\
P_{1311}^{\mu\nu\rho\sigma} = P_{2122}^{\mu\nu\rho\sigma} = 2P^{\mu\nu\rho\sigma}, &\qquad P_{3233}^{\mu\nu\rho\sigma} = -2P^{\mu\nu\rho\sigma}, \\
P_{1123}^{\mu\nu\rho\sigma} = 3P^{\mu\nu\rho\sigma}, \qquad P_{2231}^{\mu\nu\rho\sigma} = P_{3312}^{\mu\nu\rho\sigma} &= -P^{\mu\nu\rho\sigma}, \qquad P_{1233}^{\mu\nu\rho\sigma} = 4P^{\mu\nu\rho\sigma}.
\end{aligned} \tag{4.57}$$

Here we have the generic form

$$P^{\mu\nu\rho\sigma}(\partial) = \sum_{p,q=1}^{2} a^{\{pq\}}\ A_{pq}^{\mu\nu\rho\sigma}(\partial) + (\sum_{p,q=1}^{2} b^{\{pq\}}\ \partial_p \cdot \partial_q + c)\ A^{\mu\nu\rho\sigma} \tag{4.58}$$

where

$$A_{pq}^{\mu\nu\rho\sigma}(\partial) = \eta^{\mu\rho}\ \partial_p^\nu \partial_q^\sigma + \eta^{\mu\sigma}\ \partial_p^\nu \partial_q^\rho + \eta^{\nu\rho}\ \partial_p^\mu \partial_q^\sigma + \eta^{\nu\sigma}\ \partial_p^\mu \partial_q^\rho + \eta^{\mu\nu}\ \partial_p^\rho \partial_q^\sigma + \eta^{\rho\sigma}\ \partial_p^\mu \partial_q^\nu \tag{4.59}$$

and

$$A^{\mu\nu\rho\sigma} = \eta^{\mu\nu}\ \eta^{\rho\sigma} + \eta^{\mu\rho}\ \eta^{\nu\sigma} + \eta^{\mu\sigma}\ \eta^{\nu\rho}. \tag{4.60}$$

The causal splitting is preserved by the redefinitions:

$$(\mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} \to (\mathcal{D}_i^\mu \mathcal{D}_j^\nu \mathcal{D}_k^\rho \mathcal{D}_l^\sigma d)^{adv(ret)} + Q^{\mu\nu\rho\sigma}(\partial)\delta \tag{4.61}$$

where the polynomial $Q^{\mu\nu\rho\sigma}$ has the same structure as $P^{\mu\nu\rho\sigma}$.

Proof: The idea of the proof is the same as in the previous theorem, only the computations are more difficult. The result of the causal splitting is given by the relations from the statement with the polynomials $P^{\mu\nu\rho\sigma}_{ijkl}$, Lorentz covariant and constrained by the symmetry properties:

$$P^{\mu\nu\rho\sigma}_{ijkl} = P^{\mu\rho\nu\sigma}_{ikjl} = P^{\mu\nu\sigma\rho}_{ijlk} = P^{\mu\sigma\rho\nu}_{ilkj} \tag{4.62}$$

and the restriction $\deg(P^{\mu\nu\rho\sigma}_{ijkl}) \leq 2$. Now we have as above the consistency relations:

$$\begin{aligned}
P^{\mu\nu\rho\sigma}_{11kl} - P^{\mu\nu\rho\sigma}_{13kl} &= P^{\nu\mu\rho\sigma}_{23kl} - P^{\nu\mu\rho\sigma}_{22kl}\\
P^{\mu\nu\rho\sigma}_{22kl} - P^{\mu\nu\rho\sigma}_{21kl} &= P^{\nu\mu\rho\sigma}_{31kl} - P^{\nu\mu\rho\sigma}_{33kl}\\
P^{\mu\nu\rho\sigma}_{33kl} - P^{\mu\nu\rho\sigma}_{32kl} &= P^{\nu\mu\rho\sigma}_{12kl} - P^{\nu\mu\rho\sigma}_{11kl}\\
P^{\mu\nu\rho\sigma}_{13kl} - P^{\mu\nu\rho\sigma}_{12kl} &= \mu \leftrightarrow \nu\\
P^{\mu\nu\rho\sigma}_{21kl} - P^{\mu\nu\rho\sigma}_{23kl} &= \mu \leftrightarrow \nu\\
P^{\mu\nu\rho\sigma}_{32kl} - P^{\mu\nu\rho\sigma}_{31kl} &= \mu \leftrightarrow \nu.
\end{aligned} \tag{4.63}$$

We can make the redefinitions

$$(\mathcal{D}^{\mu}_{i}\mathcal{D}^{\nu}_{j}\mathcal{D}^{\rho}_{k}\mathcal{D}^{\sigma}_{l} d)^{adv(ret)} \rightarrow (\mathcal{D}^{\mu}_{i}\mathcal{D}^{\nu}_{j}\mathcal{D}^{\rho}_{k}\mathcal{D}^{\sigma}_{l} d)^{adv(ret)} + Q^{\mu\nu\rho\sigma}_{ijkl}(\partial)\ \delta \tag{4.64}$$

with $Q^{\mu\nu\rho\sigma}_{ijkl}$ polynomials in the partial derivatives of degree $\deg(Q^{\mu\nu\rho\sigma}_{ijkl}) \leq 2$, Lorentz covariant and with the symmetry property

$$Q^{\mu\nu\rho\sigma}_{ijkl} = Q^{\nu\mu\rho\sigma}_{jikl} = Q^{\mu\rho\nu\sigma}_{ikjl} = Q^{\mu\nu\sigma\rho}_{ijlk}. \tag{4.65}$$

In this way we do not disturb the order of the singularity and the causal support properties. This redefinitions change the anomalies according to

$$\begin{aligned}
P^{\mu\nu\rho\sigma}_{1jkl} &\rightarrow P^{\mu\nu\rho}_{1jkl} - Q^{\mu\nu\rho\sigma}_{3jkl} + Q^{\mu\nu\rho\sigma}_{2jkl}\\
P^{\mu\nu\rho\sigma}_{2jkl} &\rightarrow P^{\mu\nu\rho\sigma}_{2jkl} - Q^{\mu\nu\rho\sigma}_{1jkl} + Q^{\mu\nu\rho\sigma}_{3jkl}\\
P^{\mu\nu\rho\sigma}_{3jkl} &\rightarrow P^{\mu\nu\rho\sigma}_{3jkl} - Q^{\mu\nu\rho\sigma}_{2jkl} + Q^{\mu\nu\rho\sigma}_{1jkl}.
\end{aligned} \tag{4.66}$$

One can use these redefinitions to make null some of the the anomalies:

$$\begin{aligned}
P^{\mu\nu\rho\sigma}_{1222} = P^{\mu\nu\rho\sigma}_{1333} = P^{\mu\nu\rho\sigma}_{2111} = P^{\mu\nu\rho\sigma}_{2333} = P^{\mu\nu\rho\sigma}_{3111} = P^{\mu\nu\rho\sigma}_{3222} &= 0\\
P^{\mu\nu\rho\sigma}_{1323} = P^{\mu\nu\rho\sigma}_{2133} = P^{\mu\nu\rho\sigma}_{3211} &= 0\\
P^{\mu\nu\rho\sigma}_{1112} = P^{\mu\nu\rho\sigma}_{2223} = P^{\mu\nu\rho\sigma}_{3331} &= 0
\end{aligned} \tag{4.67}$$

and we are left with some redefinition freedom

$$\begin{aligned}
Q^{\mu\nu\rho\sigma}_{2111} = Q^{\mu\nu\rho\sigma}_{3111} &= Q^{\mu\nu\rho\sigma}_{1111}\\
Q^{\mu\nu\rho\sigma}_{1222} = Q^{\mu\nu\rho\sigma}_{3222} &= Q^{\mu\nu\rho\sigma}_{2222}\\
Q^{\mu\nu\rho\sigma}_{1333} = Q^{\mu\nu\rho\sigma}_{2333} &= Q^{\mu\nu\rho\sigma}_{3333}\\
Q^{\mu\nu\rho\sigma}_{3322} = Q^{\mu\nu\rho\sigma}_{2222}, \quad Q^{\mu\nu\rho\sigma}_{1133} = Q^{\mu\nu\rho\sigma}_{3333}, \quad Q^{\mu\nu\rho\sigma}_{2211} &= Q^{\mu\nu\rho\sigma}_{1111},\\
Q^{\mu\nu\rho\sigma}_{3112} = Q^{\mu\nu\rho\sigma}_{1111}, \quad Q^{\mu\nu\rho\sigma}_{1223} = Q^{\mu\nu\rho\sigma}_{2222}, \quad Q^{\mu\nu\rho\sigma}_{2331} &= Q^{\mu\nu\rho\sigma}_{3333}
\end{aligned} \tag{4.68}$$

which we will use later.

Proceeding as at the previous theorem we can use the consistency equations to obtain the generic forms

$$
\begin{aligned}
P_{1233}^{\mu\nu\rho\sigma} &= \sum_{p,q=1}^{2} a_1^{pq}\ A_{pq}^{\mu\nu\rho\sigma} + \sum_{p,q=1}^{2} a_2^{pq}\ A^{\mu\nu\rho\sigma}\ \partial_p \cdot \partial_q + a_3\ A^{\mu\nu\rho\sigma} \\
P_{2311}^{\mu\nu\rho\sigma} &= \sum_{p,q=1}^{2} b_1^{pq}\ A_{pq}^{\mu\nu\rho\sigma} + \sum_{p,q=1}^{2} b_2^{pq}\ A^{\mu\nu\rho\sigma}\ \partial_p \cdot \partial_q + b_3\ A^{\mu\nu\rho\sigma} \\
P_{3122}^{\mu\nu\rho\sigma} &= \sum_{p,q=1}^{2} c_1^{pq}\ A_{pq}^{\mu\nu\rho\sigma} + \sum_{p,q=1}^{2} c_2^{pq}\ A^{\mu\nu\rho\sigma}\ \partial_p \cdot \partial_q + c_3\ A^{\mu\nu\rho\sigma}
\end{aligned}
\tag{4.69}
$$

where we use the notations from the statement and the matrices $a_j^{pq},\ b_j^{pq}\ c_j^{pq},\ j = 1,2$ are symmetric. If we use the rest of the consistency equations and the redefinitions (4.68) we can make $b_j^{\cdot\cdot} = c_j^{\cdot\cdot} = 0$ and we finally end with expressions from the statement. ■

As before we show that in fact the anomaly from the preceding theorem is null.

Theorem 4.6 *The causal splitting can be done in such a way that*

$$
P_{ijkl}^{\mu\nu\rho\sigma} = 0. \tag{4.70}
$$

Proof: We consider in the first relation from the statement of the previous theorem $j = 3, k = l = 1$ and contract with $\eta_{\rho\sigma}$. The result is

$$
\frac{\partial}{\partial x_\mu}(\mathcal{D}_3^\nu \mathcal{K}_1 d_{123})^{adv(ret)} = (\mathcal{D}_3^\mu \mathcal{D}_3^\nu \mathcal{K}_1 d)^{adv(ret)} - (\mathcal{D}_2^\mu \mathcal{D}_3^\nu \mathcal{K}_1 d)^{adv(ret)} + \eta_{\rho\sigma}\ P_{1311}^{\mu\nu\rho\sigma}. \tag{4.71}
$$

But it easy to prove that

$$
\begin{aligned}
\mathcal{D}_3^\nu \mathcal{K}_1 d_{123}(x,y,z) &= 2\ \delta(y-z)\ \mathcal{D}_3^\nu d_{23}(x-y) - m_1^2\ \mathcal{D}_3^\nu d_{123}(x,y,z) \\
\mathcal{D}_2^\mu \mathcal{D}_3^\nu \mathcal{K}_1 d_{123}(x,y,z) &= -2\ \delta(y-z)\ \mathcal{D}_2^\mu \mathcal{D}_3^\nu d_{23}(x-y) - m_1^2\ \mathcal{D}_2^\mu \mathcal{D}_3^\nu d_{123}(x,y,z) \\
\mathcal{D}_3^\mu \mathcal{D}_3^\nu \mathcal{K}_1 d_{123}(x,y,z) &= 2\ \delta(y-z)\ \mathcal{D}_3^\mu \mathcal{D}_3^\nu d_{23}(x-y) - m_1^2\ \mathcal{D}_3^\mu \mathcal{D}_3^\nu d_{123}(x,y,z)
\end{aligned}
\tag{4.72}
$$

so the preceding relation becomes

$$
\begin{aligned}
2\ \delta(y-z)\ \frac{\partial}{\partial x_\mu}(\mathcal{D}_3^\nu d_{23})^{adv(ret)}(x-y) - m_1^2\ \frac{\partial}{\partial x_\mu}(\mathcal{D}_3^\nu d_{123})^{adv(ret)}(x,y,z) = \\
2\ \delta(y-z)\ (\mathcal{D}_3^\mu \mathcal{D}_3^\nu d_{23})^{adv(ret)}(x-y) - m_1^2\ (\mathcal{D}_3^\mu \mathcal{D}_3^\nu d_{123})^{adv(ret)}(x,y,z) \\
+2\ \delta(y-z)\ (\mathcal{D}_2^\mu \mathcal{D}_3^\nu d_{23})^{adv(ret)}(x-y) + m_1^2\ (\mathcal{D}_2^\mu \mathcal{D}_3^\nu d_{123})^{adv(ret)}(x,y,z) \\
+\eta_{\rho\sigma}\ P_{1e11}^{\mu\nu\rho\sigma}(x,y,z).
\end{aligned}
\tag{4.73}
$$

Now from theorem 3.1 we have

$$
\frac{\partial}{\partial x_\mu}(\mathcal{D}_3^\nu d_{23})^{adv(ret)} = (\mathcal{D}_2^\mu \mathcal{D}_3^\nu d_{23})^{adv(ret)} + (\mathcal{D}_3^\mu \mathcal{D}_3^\nu d_{23})^{adv(ret)} \tag{4.74}
$$

so if we use the preceding theorem also we are left with

$$\eta_{\rho\sigma}\; P_{1311}^{\mu\nu\rho\sigma}(x,y,z) = 0. \tag{4.75}$$

If we insert the explicit form of the anomaly from the preceding theorem then we get

$$P_{ijkl}^{\mu\nu\rho\sigma} = 0. \tag{4.76}$$

Now we can use the preceding results to prove:

Theorem 4.7 *We consider as before the Yang-Mills case. Then the one-loop triangle contributions do not produce anomalies in the third order perturbation theory.*

Proof: The one-loop triangle contribution to the causal commutators $D^{I,J,K}$ are sums of contributions of the type

$$d^{(3)}_{D_1,D_2,D_3}\; W(x,y,z), \quad \mathcal{D}^{\mu}_{j} d^{(3)}_{D_1,D_2,D_3}\; W(x,y,z), \quad \mathcal{D}^{\mu}_{j}\mathcal{D}^{\nu}_{k} d^{(3)}_{D_1,D_2,D_3}\; W(x,y,z),$$
$$\mathcal{D}^{\mu}_{j}\mathcal{D}^{\nu}_{k}\mathcal{D}^{\rho}_{l} d^{(3)}_{D_1,D_2,D_3}\; W(x,y,z) \tag{4.77}$$

where W are Wick monomials. These expressions come with various numerical coefficients t (which are in fact Lorentz tensors). This follows from the limitations of the Yang-Mills model: we have at most a derivative in the interaction Lagrangian, so we have at most three derivatives $\mathcal{D}$ on d. The gauge invariance (1.7) gives immediately

$$d_Q D^{I,J,K} = i\frac{\partial}{\partial x^{\mu}} D^{I\mu,J,K} + (-1)^{|I|}\frac{\partial}{\partial y^{\mu}} D^{I,J\mu,K} + (-1)^{|I|+|J|}\frac{\partial}{\partial z^{\mu}} D^{I,J,K\mu} \tag{4.78}$$

and in particular for the one-loop triangle contribution

$$d_Q D^{I,J,K}_{(1)} = i\frac{\partial}{\partial x^{\mu}} D^{I\mu,J,K}_{(1)} + (-1)^{|I|}\frac{\partial}{\partial y^{\mu}} D^{I,J\mu,K}_{(1)} + (-1)^{|I|+|J|}\frac{\partial}{\partial z^{\mu}} D^{I,J,K\mu}_{(1)}. \tag{4.79}$$

If we substitute in this relation the expression of the one-loop contributions as a sum of expressions of the type (4.77) with various numerical coefficients, then we will obtain some identities between the coefficients t **modulo** the identities (4.7), (4.10), (4.19) and (4.55). So if we can causally split these relation without anomalies we will obtain from the preceding relation

$$d_Q A^{I,J,K}_{(1)} = i\frac{\partial}{\partial x^{\mu}} A^{I\mu,J,K}_{(1)} + (-1)^{|I|}\frac{\partial}{\partial y^{\mu}} A^{I,J\mu,K}_{(1)} + (-1)^{|I|+|J|}\frac{\partial}{\partial z^{\mu}} A^{I,J,K\mu}_{(1)}$$
$$d_Q R^{I,J,K}_{(1)} = i\frac{\partial}{\partial x^{\mu}} R^{I\mu,J,K}_{(1)} + (-1)^{|I|}\frac{\partial}{\partial y^{\mu}} R^{I,J\mu,K}_{(1)} + (-1)^{|I|+|J|}\frac{\partial}{\partial z^{\mu}} R^{I,J,K\mu}_{(1)} \tag{4.80}$$

so finally

$$d_Q T^{I,J,K}_{(1)} = i\frac{\partial}{\partial x^{\mu}} T^{I\mu,J,K}_{(1)} + (-1)^{|I|}\frac{\partial}{\partial y^{\mu}} T^{I,J\mu,K}_{(1)} + (-1)^{|I|+|J|}\frac{\partial}{\partial z^{\mu}} T^{I,J,K\mu}_{(1)} \tag{4.81}$$

i.e. gauge invariance for the one-loop triangle contributions. But we just have proved in the preceding theorems that such a causal splitting can be done. ■

Remark 4.8 *If we consider gravity, then we must extend the list (3.40) up to six derivatives $\mathcal{D}$ and we must investigate accordingly if the causal splitting is possible without anomalies.*

We can cover the two loop contributions if we observe that the corresponding distributions can be obtained if in $d_{123} = d_{D_1,D_2,D_3}$ we replace $D_3 \to d_{34} = d_{D_3,D_4}$. ■

In the third order of perturbation theory other causal distributions can appear. These causal distributions are associated to the one-particle reducible graphs.

$$\begin{aligned} d^{(1)}_{D_1,D_2}(x,y,z) \equiv & \bar{D}^F_1(x-y)D_2(z-x) - D_1(x-y)D^F_2(z-x) \\ & + D^{(-)}_1(x-y)D^{(+)}_2(z-x) - D^{(+)}_1(x-y)D^{(-)}_2(z-x)] \\ d^{(2)}_{D_1,D_2}(x,y,z) \equiv & -\bar{D}^F_1(x-y)D_2(y-z) + D_1(x-y)D^F_2(y-z) \\ & + D^{(+)}_1(x-y)D^{(-)}_2(y-z) - D^{(-)}_1(x-y)D^{(+)}_2(y-z)] \\ d^{(3)}_{D_1,D_2}(x,y,z) \equiv & D^F_1(z-x)D_2(y-z) - D_1(z-x)D^F_2(y-z) \\ & + D^{(-)}_1(z-x)D^{(+)}_2(y-z) - D^{(+)}_1(z-x)D^{(-)}_2(y-z)] \end{aligned} \tag{4.82}$$

The causal support properties follow from the alternative formulas

$$\begin{aligned} d^{(1)}_{D_1,D_2}(x,y,z) &= D^{\rm ret}_1(x-y)D^{\rm ret}_2(z-x) - D^{\rm adv}_1(x-y)D^{\rm adv}_2(z-x) \\ d^{(2)}_{D_1,D_2}(x,y,z) &= D^{\rm ret}_1(y-x)D^{\rm ret}_2(z-y) - D^{\rm adv}_1(y-x)D^{\rm adv}_2(z-y) \\ d^{(3)}_{D_1,D_2}(x,y,z) &= D^{\rm ret}_1(z-x)D^{\rm ret}_2(y-z) - D^{\rm adv}_1(z-x)D^{\rm adv}_2(y-z). \end{aligned} \tag{4.83}$$

The order of singularity of these distributions is again $\omega = -2$. We can define associated distributions as before if we replace $D_1 \mapsto \partial_\alpha D_1$, etc.

$$\begin{aligned} \mathcal{D}^2_\alpha d^{(1)}_{D_1,D_2} &= d^{(1)}_{D_1,\partial_\alpha D_2}, \qquad & \mathcal{D}^3_\alpha d^{(1)}_{D_1,D_2} &= d^{(1)}_{\partial_\alpha D_1,D_2}, \\ \mathcal{D}^1_\alpha d^{(2)}_{D_1,D_2} &= d^{(2)}_{D_1,\partial_\alpha D_2}, \qquad & \mathcal{D}^3_\alpha d^{(2)}_{D_1,D_2} &= d^{(2)}_{\partial_\alpha D_1,D_2}, \\ \mathcal{D}^3_\alpha d^{(3)}_{D_1,D_2} &= d^{(3)}_{D_1,\partial_\alpha D_2}, \qquad & \mathcal{D}^2_\alpha d^{(3)}_{D_1,D_2} &= d^{(3)}_{\partial_\alpha D_1,D_2}. \end{aligned} \tag{4.84}$$

As before we have

$$\begin{aligned} & \frac{\partial}{\partial x^\alpha} d^{(1)} = (\mathcal{D}^3_\alpha - \mathcal{D}^2_\alpha) d^{(1)}, \qquad \frac{\partial}{\partial y^\alpha} d^{(1)} = -\mathcal{D}^3_\alpha d^{(1)} \qquad \frac{\partial}{\partial z^\alpha} d^{(1)} = \mathcal{D}^2_\alpha d^{(1)} \\ & \frac{\partial}{\partial x^\alpha} d^{(2)} = \mathcal{D}^3_\alpha d^{(2)}, \qquad \frac{\partial}{\partial y^\alpha} d^{(2)} = (\mathcal{D}^1_\alpha - \mathcal{D}^3_\alpha) d^{(2)} \qquad \frac{\partial}{\partial z^\alpha} d^{(2)} = -\mathcal{D}^1_\alpha d^{(2)} \\ & \frac{\partial}{\partial x^\alpha} d^{(3)} = -\mathcal{D}^2_\alpha d^{(3)}, \qquad \frac{\partial}{\partial y^\alpha} d^{(3)} = \mathcal{D}^1_\alpha d^{(3)} \qquad \frac{\partial}{\partial z^\alpha} d^{(3)} = (\mathcal{D}^2_\alpha - \mathcal{D}^1_\alpha) d^{(3)}. \end{aligned} \tag{4.85}$$

Again we must investigate if relations of this type can be causally split. But this is obvious if we take the causal split to be

$$\begin{aligned} d^{(1)adv}_{D_1,D_2}(x,y,z) &= D^{\rm ret}_1(x-y)D^{\rm ret}_2(z-x), \quad & d^{(1)ret}_{D_1,D_2}(x,y,z) &= D^{\rm adv}_1(x-y)D^{\rm adv}_2(z-x) \\ d^{(2)adv}_{D_1,D_2}(x,y,z) &= D^{\rm ret}_1(y-x)D^{\rm ret}_2(z-y), \quad & d^{(2)ret}_{D_1,D_2}(x,y,z) &= D^{\rm adv}_1(y-x)D^{\rm adv}_2(z-y) \\ d^{(3)adv}_{D_1,D_2}(x,y,z) &= D^{\rm ret}_1(z-x)D^{\rm ret}_2(y-z), \quad & d^{(3)ret}_{D_1,D_2}(x,y,z) &= D^{\rm adv}_1(z-x)D^{\rm adv}_2(y-z) \end{aligned} \tag{4.86}$$

and similar relations for the associated distributions $\mathcal{D}^2_\alpha d^{(1)}_{D_1,D_2}$, etc.

In the functional formalism it is asserted that the axial anomaly follows from gauge invariance. We have proved that in the causal formalism this is not true. In the next Section we will offer an alternative explanation for the existence of the axial anomaly.

5 Super-Renormalizability and the Axial Anomaly

We have advocated in previous papers that one should impose the condition that the chronological products are in coboundaries up to super-renormalizable terms. We will show here that the axial anomaly must be null if we impose this condition. It can be proved that the odd-part of the causal commutators is given by

$$\begin{aligned}
D^{[\mu][\nu][\rho]}_{\rm odd}(x,y,z) &= \frac{1}{2}\ Tr(\gamma_5\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\alpha\gamma^\rho)\ \mathcal{D}_{1\alpha}\mathcal{D}_{2\beta}\mathcal{D}_{3\sigma}d(x,y,z)\ A_{abc}\ u_a(x)\ u_b(y)\ u_c(z) \\
&\qquad + {\rm super-renormalizable\ terms} \\
D^{[\mu][\nu]\emptyset}_{\rm odd}(x,y,z) &= \frac{1}{2}\ Tr(\gamma_5\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\alpha\gamma^\rho)\ \mathcal{D}_{1\alpha}\mathcal{D}_{2\beta}\mathcal{D}_{3\sigma}d(x,y,z)\ A_{abc}\ u_a(x)\ u_b(y)\ v_{c\rho}(z) \\
&\qquad + {\rm super-renormalizable\ terms} \\
D^{\emptyset\emptyset[\mu]}_{\rm odd}(x,y,z) &= -\frac{1}{2}\ Tr(\gamma_5\gamma^\beta\gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\alpha\gamma^\mu)\ \mathcal{D}_{1\alpha}\mathcal{D}_{2\beta}\mathcal{D}_{3\sigma}d(x,y,z)\ A_{abc}\ v_{a\nu}(x)\ v_{b\rho}(y)\ u_c(z) \\
&\qquad + {\rm super-renormalizable\ terms} \\
D^{\emptyset\emptyset\emptyset}_{\rm odd}(x,y,z) &= \frac{1}{2}\ Tr(\gamma_5\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\alpha\gamma^\rho)\ \mathcal{D}_{1\alpha}\mathcal{D}_{2\beta}\mathcal{D}_{3\sigma}d(x,y,z)\ A_{abc}\ v_{a\mu}(x)\ v_{b\nu}(y)\ v_{c\rho}(z) \\
&\qquad + {\rm super-renormalizable\ terms} \qquad (5.1)
\end{aligned}$$

Here $d = d_{D,D,D}$ where $D = D_0$ and

$$A_{abc} \equiv \sum_\epsilon\ \epsilon\ Tr(\{t^\epsilon_a, t^\epsilon_b\}t^\epsilon_c) \tag{5.2}$$

is a symmetric tensor build from the representation t^ϵ_a appearing in the interaction Lagrangian

$$v^\mu_a\ j_{a\mu} \tag{5.3}$$

between the Yang-Mills fields v^μ_a and the current

$$j_{a\mu} = \sum_\epsilon\ \bar{\Psi}t^\epsilon_a \otimes \gamma_\mu\gamma_\epsilon\Psi \tag{5.4}$$

associated to the Dirac fields $\Psi = (\Psi_A)_{A=1,\ldots,N}$ which are describing the matter; here $\gamma_\epsilon = \frac{1}{2}(I + \epsilon\ \gamma_5),\ \epsilon = \pm$ select the two chiralities.

We need an explicit expression for the trace $Tr(\gamma_5\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\alpha\gamma^\rho)$. From Lorentz invariance, it must be a sum of expressions of the type $\eta^{\mu\nu}\ \epsilon^{\rho\sigma\alpha\beta}$; there are 15 combinations of this type but not all of them are linear independent because of the identity

$$\eta^{\sigma\nu}\ \epsilon^{\mu\nu\alpha\rho} - \eta^{\sigma\mu}\ \epsilon^{\beta\mu\alpha\rho} + \eta^{\sigma\nu}\ \epsilon^{\beta\mu\alpha\rho} - \eta^{\sigma\alpha}\ \epsilon^{\beta\mu\nu\rho} + \eta^{\sigma\rho}\ \epsilon^{\beta\mu\nu\alpha} = 0; \tag{5.5}$$

indeed the expression is completely antisymmetric in the five indexes $\beta,\mu,\nu,\alpha,\rho$ so it must be null. We take the generic form

$$\begin{aligned}
&Tr(\gamma_5\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\alpha\gamma^\rho) = \\
a_1\ \eta^{\mu\nu}\ \epsilon^{\rho\sigma\alpha\beta} + a_2\ \eta^{\mu\rho}\ \epsilon^{\nu\sigma\alpha\beta} + a_3\ \eta^{\nu\rho}\ \epsilon^{\mu\sigma\alpha\beta} + a_4\ \eta^{\mu\alpha}\ \epsilon^{\nu\rho\sigma\beta} + a_5\ \eta^{\mu\beta}\ \epsilon^{\nu\rho\sigma\alpha} \\
+a_6\ \eta^{\nu\alpha}\ \epsilon^{\mu\sigma\sigma\beta} + a_7\ \eta^{\nu\beta}\ \epsilon^{\mu\rho\sigma\beta} + a_8\ \eta^{\rho\alpha}\ \epsilon^{\mu\nu\sigma\beta} + a_9\ \eta^{\rho\beta}\ \epsilon^{\mu\nu\sigma\alpha} + a_{10}\ \eta^{\alpha\beta}\ \epsilon^{\mu\nu\rho\sigma}
\end{aligned} \tag{5.6}$$

because the other 5 possibilities can be eliminated due to the identity (5.5). Now we observe that

$$Tr(\gamma_5\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\alpha\gamma^\rho) + \mu \leftrightarrow \beta = 2\eta^{\mu\beta}\ Tr(\gamma_5\gamma^\sigma\gamma^\nu\gamma^\alpha\gamma^\rho) = 8\eta^{\mu\beta}\ \epsilon^{\sigma\nu\alpha\rho} \tag{5.7}$$

and this leads to

$$a_7 = a_1, \quad a_9 = a_2, \quad a_{10} = a_4, \quad a_5 = -4. \tag{5.8}$$

We have three more identities of this type if we symmetrise in $\nu \leftrightarrow \alpha,\ \alpha \leftrightarrow \rho,\ \rho \leftrightarrow \beta$ and we obtain in the end

$$\begin{aligned} a_2 = a_3 = a_4 = a_6 = a_6 = a_8 = a_9 = a_{10} = 4 \\ a_1 = a_5 = a_7 = -4. \end{aligned} \tag{5.9}$$

It is easy to see that the expression so obtained is invariant with respect to the permutations

$$\mu \to \nu \to \rho \to \mu, \quad \beta \to \sigma \to \alpha \to \beta. \tag{5.10}$$

Also it is antisymmetric with respect to the change

$$\mu \leftrightarrow \nu, \quad \alpha \leftrightarrow \beta. \tag{5.11}$$

It is also antisymmetric with respect to the change

$$\nu \leftrightarrow \rho, \quad \beta \leftrightarrow \sigma \tag{5.12}$$

but only if we use the identity (5.5). However, we can rewrite the trace in an equivalent form namely

$$\begin{aligned} Tr(\gamma_5\gamma^\beta\gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\alpha\gamma^\rho) = \\ \frac{4}{3}\ (\eta^{\mu\nu}\ \epsilon^{\rho\sigma\alpha\beta} + \eta^{\mu\rho}\ \epsilon^{\nu\sigma\alpha\beta} + \eta^{\nu\rho}\ \epsilon^{\mu\sigma\alpha\beta}) \\ +\frac{4}{3}\ (\eta^{\alpha\beta}\ \epsilon^{\mu\nu\rho\sigma} - \eta^{\alpha\sigma}\ \epsilon^{\mu\rho\nu\beta} + \eta^{\beta\sigma}\ \epsilon^{\nu\rho\mu\alpha}) \\ +\frac{8}{3}\ (\eta^{\nu\alpha}\ \epsilon^{\mu\rho\sigma\beta} - \eta^{\mu\beta}\ \epsilon^{\nu\rho\sigma\alpha} - \eta^{\rho\alpha}\ \epsilon^{\mu\nu\beta\sigma} - \eta^{\mu\sigma}\ \epsilon^{\nu\rho\beta\alpha} + \eta^{\nu\sigma}\ \epsilon^{\mu\rho\alpha\beta} - \eta^{\rho\beta}\ \epsilon^{\mu\nu\alpha\sigma}) \end{aligned} \tag{5.13}$$

and one can check that all three contributions are antisymmetric with respect to

$$\mu \leftrightarrow \nu, \quad \alpha \leftrightarrow \beta \tag{5.14}$$

and

$$\nu \leftrightarrow \rho, \quad \beta \leftrightarrow \sigma. \tag{5.15}$$

If we use this antysymmetry property in (5.1) then we obtain that the expression $D^{[\mu][\nu][\rho]}_{\rm odd}(x,y,z)$ changes the sign if we make the transformations

$$x \leftrightarrow y,\ \mu \leftrightarrow \nu,\ \mathcal{D}_1 \to -\mathcal{D}_2\ \mathcal{D}_2 \to -\mathcal{D}_1\ \mathcal{D}_3 \to -\mathcal{D}_3 \tag{5.16}$$

and

$$y \leftrightarrow z,\ \nu \leftrightarrow \rho,\ \mathcal{D}_1 \to -\mathcal{D}_1\ \mathcal{D}_2 \to -\mathcal{D}_3\ \mathcal{D}_3 \to -\mathcal{D}_2 \tag{5.17}$$

and one can prove that this true also from some abstract arguments presented in [8]. Now we prove

Theorem 5.1 *The expression $D^{IJK}_{\rm odd}$ is a coboundary if and only if*

$$A_{abc} = 0. \tag{5.18}$$

Proof: We consider an arbitrary cochain cochain B^{IJK} and impose

$$D_{\rm odd}(x,y,z) = (d_Q + i\delta) B \tag{5.19}$$

In particular we must have in top ghost number

$$\begin{aligned} D^{[\mu][\nu][\rho]}_{\rm odd}(x,y,z) = d_Q B^{[\mu][\nu][\rho]} + i\ \left(\frac{\partial}{\partial x^\sigma} B^{[\mu\sigma][\nu][\rho]} - \frac{\partial}{\partial y^\sigma} B^{[\mu][\nu\sigma][\rho]} + \frac{\partial}{\partial z^\sigma} B^{[\mu][\nu][\rho\sigma]} \right) \\ d_Q B^{[\mu\nu][\rho]\emptyset} + i\ \left(\frac{\partial}{\partial y^\sigma} B^{[\mu\nu][\rho\sigma]\emptyset} - \frac{\partial}{\partial z^\sigma} B^{[\mu\nu][\rho][\sigma]} \right) = 0. \end{aligned} \tag{5.20}$$

The generic form of the cochains is

$$\begin{aligned} B^{[\mu][\nu][\rho\sigma]}(x,y,z) &= \sum a^{(j)}_{abc}\ b^{[\mu][\nu][\rho\sigma]}_j(x,y,z)\ u_a(x)\ u_b(y)\ u_c(z) \\ B^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) &= \sum b^{(j)}_{abc}\ b^{[\mu\nu][\rho\sigma]\emptyset}_j(x,y,z)\ u_a(x)\ u_b(y)\ u_c(z) \end{aligned} \tag{5.21}$$

where we impose symmetry conditions

$$\begin{aligned} B^{[\mu][\nu][\rho\sigma]}(x,y,z) &= -B^{[\nu][\mu][\rho\sigma]}(y,x,z) \\ B^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) &= B^{[\rho\sigma][\mu\nu]\emptyset}(y,x,z) \end{aligned} \tag{5.22}$$

and this leads to the following possible combinations:

$$\begin{aligned} b^{[\mu][\nu][\rho\sigma]}_1(x,y,z) &= \eta^{\mu\nu}\ \epsilon^{\rho\sigma\alpha\beta}\ (\mathcal{D}_{1\alpha}\mathcal{D}_{3\beta} + \mathcal{D}_{2\alpha}\mathcal{D}_{3\beta}) d(x,y,z) \\ b^{[\mu][\nu][\rho\sigma]}_2(x,y,z) &= (\eta^{\mu\rho}\ \epsilon^{\nu\sigma\alpha\beta}\ \mathcal{D}_{1\alpha}\mathcal{D}_{3\beta} + \eta^{\nu\rho}\ \epsilon^{\mu\sigma\alpha\beta}\ \mathcal{D}_{2\alpha}\mathcal{D}_{3\beta}) d(x,y,z) - \rho \leftrightarrow \sigma \\ b^{[\mu][\nu][\rho\sigma]}_3(x,y,z) &= (\eta^{\nu\rho}\ \epsilon^{\mu\sigma\alpha\beta}\ \mathcal{D}_{1\alpha}\mathcal{D}_{3\beta} + \eta^{\mu\rho}\ \epsilon^{\nu\sigma\alpha\beta}\ \mathcal{D}_{2\alpha}\mathcal{D}_{3\beta}) d(x,y,z) - \rho \leftrightarrow \sigma \\ b^{[\mu][\nu][\rho\sigma]}_4(x,y,z) &= (\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}^\mu_1\mathcal{D}_{3\alpha} + \epsilon^{\mu\rho\sigma\alpha}\ \mathcal{D}^\nu_2\mathcal{D}_{3\alpha}) d(x,y,z) \\ b^{[\mu][\nu][\rho\sigma]}_5(x,y,z) &= (\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}^\mu_2\mathcal{D}_{3\alpha} + \epsilon^{\mu\rho\sigma\alpha}\ \mathcal{D}^\nu_1\mathcal{D}_{3\alpha}) d(x,y,z) \\ b^{[\mu][\nu][\rho\sigma]}_6(x,y,z) &= (\epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}^\sigma_1\mathcal{D}_{3\alpha} - \epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}^\sigma_2\mathcal{D}_{3\alpha}) d(x,y,z) - \rho \leftrightarrow \sigma \\ b^{[\mu][\nu][\rho\sigma]}_7(x,y,z) &= (\epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_{[}\infty\alpha\mathcal{D}^\sigma_3 - \epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_{2\alpha}\mathcal{D}^\sigma_3) d(x,y,z) - \rho \leftrightarrow \sigma \\ b^{[\mu][\nu][\rho\sigma]}_8(x,y,z) &= (\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_{1\alpha}\mathcal{D}^\mu_3 + \epsilon^{\mu\rho\sigma\alpha}\ \mathcal{D}_{2\alpha}\mathcal{D}^\nu_3) d(x,y,z) \\ b^{[\mu][\nu][\rho\sigma]}_9(x,y,z) &= (\epsilon^{\mu\rho\sigma\alpha}\ \mathcal{D}_{1\alpha}\mathcal{D}^\nu_3 + \epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_{2\alpha}\mathcal{D}^\mu_3) d(x,y,z) \\ b^{[\mu][\nu][\rho\sigma]}_{10}(x,y,z) &= \epsilon^{\mu\rho\sigma\sigma}\ (\mathcal{D}_1 \cdot \mathcal{D}_3 - \mathcal{D}_2 \cdot \mathcal{D}_3) d(x,y,z) \\ b^{[\mu][\nu][\rho\sigma]}_{11}(x,y,z) &= (\eta^{\mu\rho}\ \epsilon^{\nu\sigma\alpha\beta}\ \mathcal{D}_{1\alpha}\mathcal{D}_{2\beta} - \eta^{\nu\rho}\ \epsilon^{\mu\sigma\alpha\beta}\ \mathcal{D}_{1\alpha}\mathcal{D}_{2\beta}) d(x,y,z) - \rho \leftrightarrow \sigma \\ b^{[\mu][\nu][\rho\sigma]}_{12}(x,y,z) &= (\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}^\mu_1\mathcal{D}_{2\alpha} + \epsilon^{\mu\rho\sigma\alpha}\ \mathcal{D}_{1\alpha}\mathcal{D}^\nu_2) d(x,y,z) \\ b^{[\mu][\nu][\rho\sigma]}_{13}(x,y,z) &= (\epsilon^{\mu\rho\sigma\alpha}\ \mathcal{D}^\nu_1\mathcal{D}_{2\alpha} + \epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_{1\alpha}\mathcal{D}^\mu_2) d(x,y,z) \end{aligned}$$

$$b_{14}^{[\mu][\nu][\rho\sigma]}(x,y,z) = (\epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_1^\sigma \mathcal{D}_{2\alpha} - \epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_{1\alpha}\mathcal{D}_2^\sigma)d(x,y,z) - \rho \leftrightarrow \sigma$$
$$b_{15}^{[\mu][\nu][\rho\sigma]}(x,y,z) = (\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_1^\mu \mathcal{D}_{1\alpha} + \epsilon^{\mu\rho\sigma\alpha}\ \mathcal{D}_2^\nu\mathcal{D}_{2\alpha})d(x,y,z)$$
$$b_{16}^{[\mu][\nu][\rho\sigma]}(x,y,z) = (\epsilon^{\mu\rho\sigma\alpha}\ \mathcal{D}_1^\nu \mathcal{D}_{1\alpha} + \epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_2^\mu\mathcal{D}_{2\alpha})d(x,y,z)$$
$$b_{17}^{[\mu][\nu][\rho\sigma]}(x,y,z) = (\epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_1^\sigma \mathcal{D}_{1\alpha} - \epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_2^\sigma\mathcal{D}_{2\alpha})d(x,y,z) - \rho \leftrightarrow \sigma$$
$$b_{18}^{[\mu][\nu][\rho\sigma]}(x,y,z) = (\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_3^\mu \mathcal{D}_{3\alpha} + \epsilon^{\mu\rho\sigma\alpha}\ \mathcal{D}_3^\nu\mathcal{D}_{3\alpha})d(x,y,z)$$
$$b_{19}^{[\mu][\nu][\rho\sigma]}(x,y,z) = \epsilon^{\mu\nu\rho\sigma}\ (\mathcal{K}_1 - \mathcal{K}_2)d(x,y,z) \tag{5.23}$$

and

$$b_1^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = (\eta^{\mu\rho}\ \epsilon^{\nu\sigma\alpha\beta} - \eta^{\nu\rho}\ \epsilon^{\mu\sigma\alpha\beta})\ (\mathcal{D}_{1\alpha}\mathcal{D}_{3\beta} + \mathcal{D}_{2\alpha}\mathcal{D}_{3\beta})d(x,y,z) - \rho \leftrightarrow \sigma$$
$$b_2^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = [(\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_1^\mu\mathcal{D}_{3\alpha} - \mu \leftrightarrow \nu) + (\epsilon^{\rho\mu\nu\alpha}\ \mathcal{D}_2^\sigma\mathcal{D}_{3\alpha} - \rho \leftrightarrow \sigma)]d(x,y,z)$$
$$b_3^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = [(\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_2^\mu\mathcal{D}_{3\alpha} - \mu \leftrightarrow \nu) + (\epsilon^{\rho\mu\nu\alpha}\ \mathcal{D}_1^\sigma\mathcal{D}_{3\alpha} - \rho \leftrightarrow \sigma)]d(x,y,z)$$
$$b_4^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = [(\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_{1\alpha}\mathcal{D}_3^\mu - \mu \leftrightarrow \nu) + (\epsilon^{\rho\mu\nu\alpha}\ \mathcal{D}_{2\alpha}\mathcal{D}_3^\rho - \rho \leftrightarrow \sigma)]d(x,y,z)$$
$$b_5^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = [(\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_{1\alpha}\mathcal{D}_3^\mu - \mu \leftrightarrow \nu) + (\epsilon^{\rho\mu\nu\alpha}\ \mathcal{D}_{2\alpha}\mathcal{D}_3^\sigma - \rho \leftrightarrow \sigma)]d(x,y,z)$$
$$b_6^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = \epsilon^{\mu\nu\rho\sigma}\ (\mathcal{D}_1\cdot\mathcal{D}_3 - \mathcal{D}_2\cdot\mathcal{D}_3)d(x,y,z)$$
$$b_7^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = [(\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_1^\mu\mathcal{D}_{2\alpha} - \mu \leftrightarrow \nu) + (\epsilon^{\rho\mu\nu\alpha}\ \mathcal{D}_{1\alpha}\mathcal{D}_2^\sigma - \rho \leftrightarrow \sigma)]d(x,y,z)$$
$$b_8^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = [(\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_{1\alpha}\mathcal{D}_2^\mu - \mu \leftrightarrow \nu) + (\epsilon^{\rho\mu\nu\alpha}\ \mathcal{D}_1^\sigma\mathcal{D}_{2\alpha} - \rho \leftrightarrow \sigma)]d(x,y,z)$$
$$b_9^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = [(\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_1^\mu\mathcal{D}_{1\alpha} - \mu \leftrightarrow \nu) + (\epsilon^{\rho\mu\nu\alpha}\ \mathcal{D}_2^\sigma\mathcal{D}_{2\alpha} - \rho \leftrightarrow \sigma)]d(x,y,z)$$
$$b_{10}^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = [(\epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_1^\sigma\mathcal{D}_{1\alpha} - \rho \leftrightarrow \sigma) + (\epsilon^{\rho\sigma\nu\alpha}\ \mathcal{D}_2^\mu\mathcal{D}_{2\alpha} - \mu \leftrightarrow \nu)]d(x,y,z)$$
$$b_{11}^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = [(\epsilon^{\nu\rho\sigma\alpha}\ \mathcal{D}_3^\mu\mathcal{D}_{3\alpha} - \mu \leftrightarrow \nu) + (\epsilon^{\rho\mu\nu\alpha}\ \mathcal{D}_3^\sigma\mathcal{D}_{3\alpha} - \rho \leftrightarrow \sigma)]d(x,y,z)$$
$$b_{12}^{[\mu\nu][\rho\sigma]\emptyset}(x,y,z) = \epsilon^{\mu\nu\rho\alpha}\ (\mathcal{K}_1 -\ \mathcal{K}_2)d(x,y,z). \tag{5.24}$$

We consider the first relation (5.20) and we obtain 10 independent combinations in the $u_a(x)\ u_b(y)\ u_c(z)$ sector. For instance the coefficient of

$$(\eta^{\mu\nu}\ \epsilon^{\rho\sigma\alpha\beta} + \eta^{\mu\rho}\ \epsilon^{\nu\sigma\alpha\beta} + \eta^{\nu\rho}\ \epsilon^{\mu\sigma\alpha\beta})\ \mathcal{D}_{1\alpha}\mathcal{D}_{2\beta}\mathcal{D}_{3\sigma}d \tag{5.25}$$

gives the identity

$$a_1 + a_2 + a_3 = \frac{2i}{3}A \tag{5.26}$$

and the coefficient of

$$\epsilon^{\mu\nu\rho\sigma}\ (\mathcal{D}_{1\sigma}\mathcal{K}_1 + \mathcal{D}_{2\sigma}\mathcal{K}_2 + \mathcal{D}_{3\sigma}\mathcal{K}_3)d \tag{5.27}$$

gives

$$a_{17} + a_{19} = 0. \tag{5.28}$$

Here we have used the simplified notation $a_{abc}^{(j)} \to a_j,\ A_{abc} \to A$.

The second equation (5.20) is more difficult to analyze because of the identity (5.5). We consider as before the sector $u_a(x)\ u_b(y)\ u_c(z)$ and have three cases:

1) The coefficients of $\mathcal{D}_j\mathcal{D}_j\mathcal{D}_j,\ j=1,2,3$ We have three independent combinations

$$\epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_{j\alpha}\mathcal{K}_j d,\ \ j=1,2,3 \tag{5.29}$$

and they produce the equations

$$\begin{aligned} a_{18}+b_{10}+b_{12}&=0\\ a_{16}+a_{19}&=0\\ b_{11}&=0. \end{aligned} \tag{5.30}$$

2) The coefficients of

$$\begin{aligned} X_1^{\mu\nu\rho}&=\epsilon^{\mu\nu\alpha\beta}\ \mathcal{D}_{j\alpha}\mathcal{D}_{l\beta}\mathcal{D}_l^\rho d, \quad X_2^{\mu\nu\rho}=\epsilon^{\mu\nu\alpha\beta}\ \mathcal{D}_{l\alpha}\mathcal{D}_j\cdot\mathcal{D}_l d,\\ X_3^{\mu\nu\rho}&=\epsilon^{\mu\nu\alpha\beta}\ \mathcal{D}_{j\alpha}\mathcal{K}_l d, \quad X_4^{\mu\nu\rho}=\epsilon^{\nu\rho\alpha\beta}\ \mathcal{D}_{j\alpha}\mathcal{D}_{l\beta}\mathcal{D}_l^\mu d-\mu\leftrightarrow\nu \end{aligned} \tag{5.31}$$

for $j\neq l$. Because of (5.5) we have the identities of the type

$$X_4^{\mu\nu\rho}=-X_1^{\mu\nu\rho}+X_2^{\mu\nu\rho}-X_3^{\mu\nu\rho} \tag{5.32}$$

so we have only 6×3 relations of this type

$$\begin{aligned} -a_1-a_3+a_7+a_8+a_{18}-b_7-b_8&=0\\ a_3-a_5-a_7-a_{10}+a_{18}&=0\\ a_3-a_7+a_9+b_7+b_8&=0 \end{aligned} \tag{5.33}$$

$$\begin{aligned} a_1+a_3+a_4+a_6+a_{15}+a_{17}-b_9-b_{10}&=0\\ -a_3-a_6-a_9-a_{10}+a_{16}-a_{17}+b_9+b_{10}&=0\\ -a_3+a_5-a_6-a_{17}-a_{19}+b_{10}+b_{12}&=0 \end{aligned} \tag{5.34}$$

$$\begin{aligned} -a_1-a_2-a_7+a_9+b_1-b_2-b_3-b_9-b_{10}&=0\\ -a_2+a_4-a_7-a_{10}+b_1-b_2+b_5+b_6-b_9-b_{10}&=0\\ a_2+a_7+a_8-b_1+b_2+b_3+b_9-b_{12}&=0 \end{aligned} \tag{5.35}$$

$$\begin{aligned} -a_{16}+a_{17}+b_1+b_4+b_5+2b_{11}&=0\\ a_{15}+a_{17}+b_1-b_3+b_4-b_6+2b_{11}&=0\\ a_{17}+a_{19}+b_1+b_4+b_5+b_{11}&=0 \end{aligned} \tag{5.36}$$

$$\begin{aligned} -a_{11}+a_{12}+a_{14}+b_9+b_{10}&=0\\ -a_{11}-a_{13}+a_{14}+b_{10}+b_{12}&=0 \end{aligned} \tag{5.37}$$

$$a_{16} - a_{17} + b_1 + b_4 + b_5 = 0$$
$$-a_{15} - a_{17} + b_1 - b_2 + b_5 + b_6 = 0$$
$$-a_{17} - a_{19} + b_1 + b_4 + b_5 = 0 \tag{5.38}$$

where we skip a relation which appears twice.

3) The coefficients of

$$Y_1^{\mu\nu\rho} = \epsilon^{\mu\nu\alpha\beta}\ \mathcal{D}_1^\rho \mathcal{D}_{2\alpha}\mathcal{D}_{3\beta}d, \quad Y_2^{\mu\nu\rho} = \epsilon^{\mu\nu\alpha\beta}\ \mathcal{D}_{1\alpha}\mathcal{D}_2^\rho \mathcal{D}_{3\beta}d, \quad Y_3^{\mu\nu\rho} = \epsilon^{\mu\nu\alpha\beta}\ \mathcal{D}_{1\alpha}\mathcal{D}_{2\beta}\mathcal{D}_3^\rho d,$$
$$Y_4^{\mu\nu\rho} = \epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_{1\alpha}\mathcal{D}_2 \cdot \mathcal{D}_3 d, \quad Y_5^{\mu\nu\rho} = \epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_{2\alpha}\mathcal{D}_1 \cdot \mathcal{D}_3 d, \quad Y_6^{\mu\nu\rho} = \epsilon^{\mu\nu\rho\alpha}\ \mathcal{D}_{3\alpha}\mathcal{D}_1 \cdot \mathcal{D}_2 d,$$
$$Y_7^{\mu\nu\rho} = \epsilon^{\mu\rho\alpha\beta}\ \mathcal{D}_1^\nu \mathcal{D}_{2\alpha}\mathcal{D}_{3\beta}d - \mu \leftrightarrow \nu, \quad Y_8^{\mu\nu\rho} = \epsilon^{\mu\rho\alpha\beta}\ \mathcal{D}_{1\alpha}\mathcal{D}_2^\nu \mathcal{D}_{3\beta}d - \mu \leftrightarrow \nu,$$
$$Y_9^{\mu\nu\rho} = \epsilon^{\mu\rho\alpha\beta}\ \mathcal{D}_{1\alpha}\mathcal{D}_{2\beta}\mathcal{D}_3^\nu d - \mu \leftrightarrow \nu, \quad Y_{10}^{\mu\nu\rho} = \eta^{\mu\rho}\ \epsilon^{\nu\alpha\beta\sigma}\ \mathcal{D}_{1\alpha}\mathcal{D}_{2\beta}\mathcal{D}_{3\sigma}d - \mu \leftrightarrow \nu. \tag{5.39}$$

Because of (5.5) we have identities

$$Y_7^{\mu\nu\rho} = Y_1^{\mu\nu\rho} + Y_5^{\mu\nu\rho} - Y_6^{\mu\nu\rho}$$
$$Y_8^{\mu\nu\rho} = Y_2^{\mu\nu\rho} + Y_4^{\mu\nu\rho} - Y_6^{\mu\nu\rho}$$
$$Y_9^{\mu\nu\rho} = Y_3^{\mu\nu\rho} + Y_4^{\mu\nu\rho} - Y_5^{\mu\nu\rho}$$
$$Y_{10}^{\mu\nu\rho} = -Y_1^{\mu\nu\rho} + Y_2^{\mu\nu\rho} - Y_3^{\mu\nu\rho} \tag{5.40}$$

we have only 6 relations in this sector

$$a_1 + a_2 + a_3 - a_{11} + a_{12} + a_{14} + b_1 - b_2 - b_3 - b_7 - b_8 = 0$$
$$a_3 - a_5 + a_6 - a_{11} - a_{13} + a_{14} + b_1 + b_4 + b_5 = 0$$
$$a_2 - a_4 - a_6 - b_3 - b_5 - b_6 - b_7 - b_8 = 0$$
$$-a_2 - a_6 - a_7 - a_{10} + a_{11} - a_{14} - b_1 - b_4 - b_5 + b_7 + b_8 = 0$$
$$a_2 + a_7 + a_8 - a_{11} - a_{13} + a_{14} + b_1 - b_2 - b_3 - b_7 - b_8 = 0 \tag{5.41}$$

where we have skipped a relation which appears previously.

Now it is not very hard to obtain from the previous equations together with (5.28) that

$$a_1 + a_2 + a_3 = 0 \tag{5.42}$$

so the equation (5.26) gives $A_{abc} = 0$ i.e. the axial anomaly is null. ■

6 Conclusions

We have proved that the loop contributions to the chronological products do not produce anomalies, at least for the second and third order of the perturbation theory. The key point was to prove that some identities involving distributions with causal support can be causally split without anomalies. We conjecture that the same is true in arbitrary order i.e. the loop contributions do not produce anomalies. Then the problem of analyzing the anomalies of gauge models is reduced to the tree sector; this can be done using the methods of [7].

From our result it follows that the axial anomaly cannot appear from pure gauge invariance consideration. To make it null we have to impose a stronger condition, namely that the chronological products are coboundaries, up to super-renormalizable terms.

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